Directional Dependence using Copulas

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Objectives

- Understanding the ideas of directional dependence, copulas, and spatial statistics
- Learn how to develop and the developing of new models
- Ultimately, apply to data

Background and basic definitions:

Directional Dependence

- Causal relationships can only be set through experiments
- Not "direction of dependence"
- Unemployment and education level

Unemployment  Education Level

- Depends on the 'way' it's looked at or introduced
  - Not limited to positive and negative dependence

Background and basic definitions:

Directional Dependence

- To better understand

http://www.upwithtrees.org/wp-content/uploads/2013/07/trees-566.jpg

http://assets.byways.org/asset_files/000/008/010/JK-view-up-tree.jpg?1258528818

Background and basic definitions:

Copulas

$$\left(X_1, X_2, \ldots, X_k\right) \Rightarrow F_{X_1}\left(x_1\right), F_{X_2}\left(x_2\right), \ldots, F_{X_k}\left(x_k\right)$$

$$F_{X_1, X_2, \ldots, X_k}\left(x_1, x_2, \ldots, x_k\right)$$

$$U_1 = F_{X_1}\left(X_1\right), U_2 = F_{X_2}\left(X_2\right), \ldots, U_k = F_{X_k}\left(x_k\right)$$

$$\left(U_1, U_2, \ldots, U_k\right) \Rightarrow C\left(u_1, u_2, \ldots, u_k\right)$$

Background and basic definitions:

Copulas

- Multivariate functions with uniform marginals
- Eliminate influence of marginals
- Used to describe the dependence between R.V.

Background and basic definitions:

Directional Dependence and Copulas

- Symmetric.
  - Example: Farlie-Gumbel-Morgenstern's family

    $C_{\theta}(u, v) = uv + \theta uv(1 - u)(1 - v)$

- If $E[V|U = w] \neq E[U|V = w]$ then we say the pair (U,V) is directionally dependent in joint behavior

Background and basic definitions:

Spatial Statistics

- 2012, Orth

$$P_{VW}(\theta) = \cos(\theta)V + \sin(\theta)W$$

$$Cor(U, P_{VW}(\theta))$$

- Now $P^1_{VW}(\alpha) = \cos(\alpha)V + \sin(\alpha)W$

$$P^2_{VW}(\beta) = -\sin(\beta)V + \cos(\beta)W$$

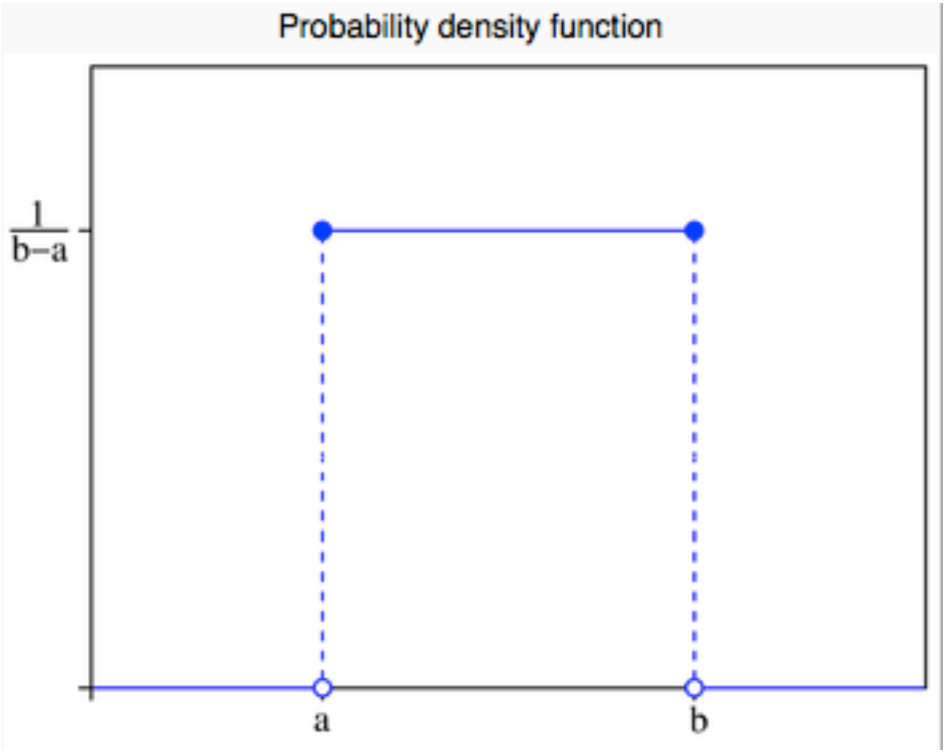
$$Cor(P^1_{VW}(\alpha), P^2_{VW}(\beta))$$

Procedure

- Start with U_1, U_2, U_3, U_4
- Uniform distribution

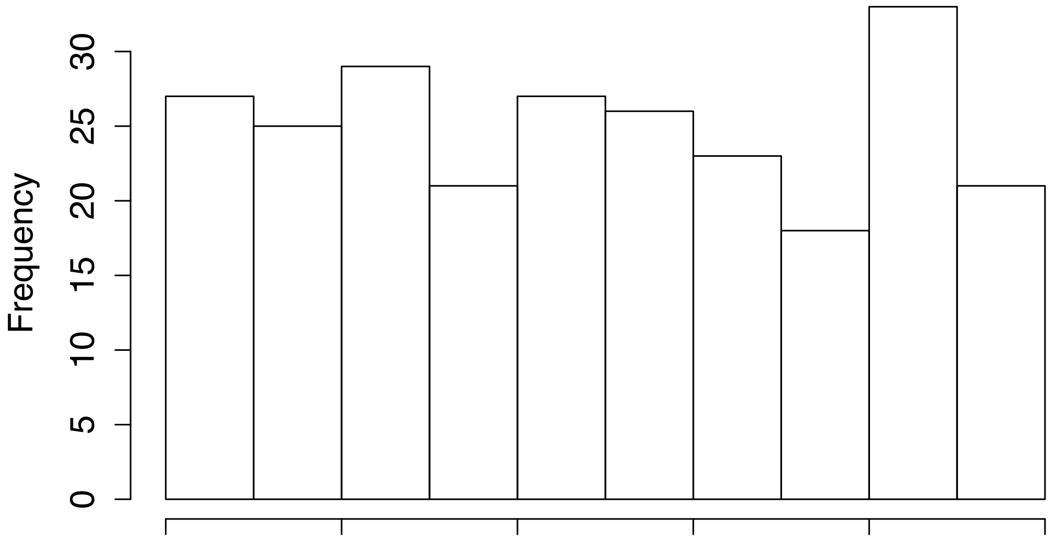
$$f(x) = \begin{cases} \dfrac{1}{b-a} & for\ a < x < b \\ 0 & otherwise \end{cases}$$

$$F(x) = \begin{cases} 0 & if\ x \le a \\ \dfrac{x-a}{b-a} & if\ a < x < b \\ 1 & if\ b \ge x \end{cases}$$

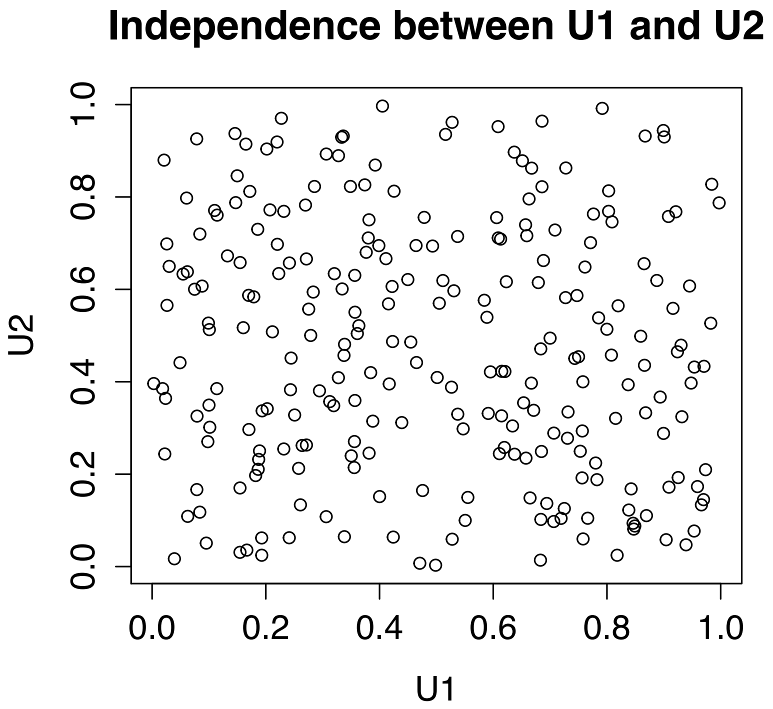


http://en.wikipedia.org/wiki/Uniform_distribution_%28continuous%29

Procedure



Frequency
0
5
10
15
20
25
30



Independence between U1 and U2
U2
0.0
0.2
0.4
0.6
0.8
1.0
U1

Procedure

$$\begin{bmatrix} \cos\alpha_1 & \sin\alpha_1 & 0 & 0 \\ -\sin\beta_1 & \cos\beta_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \end{bmatrix} = \begin{bmatrix} P^{+}_{(1,2)} \\ P^{-}_{(1,2)} \\ U_3 \\ U_4 \end{bmatrix},$$

$$P^{+}_{(1,2)} = \cos(\alpha_1)\,U1 + \sin(\alpha_1)\,U2$$

$$P^{-}_{(1,2)} = -\sin(\beta_1)\,U1 + \cos(\beta_1)\,U2$$

$$,\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\alpha_2 & \sin\alpha_2 & 0 \\ 0 & -\sin\beta_2 & \cos\beta_2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} P^{+}_{(1,2)} \\ P^{-}_{(1,2)} \\ U_3 \\ U_4 \end{bmatrix} = \begin{bmatrix} P^{+}_{(1,2)} \\ P^{-+}_{(1,2),3} \\ P^{--}_{(1,2),3} \\ U_4 \end{bmatrix}$$

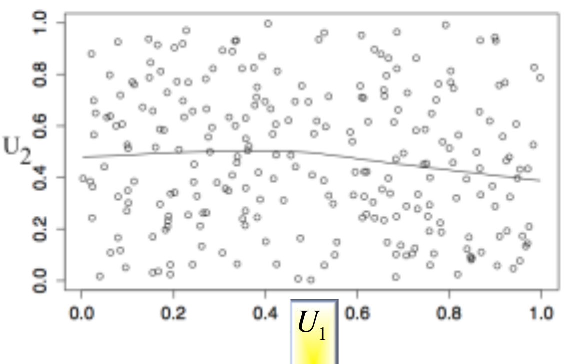
$$P^{-+}_{(1,2),3} = \cos(\alpha_2)\cos(\beta_1)\,U2 + \cos(\alpha_2)\,U3 - \cos(\alpha_2)\sin(\beta_1)\,U1$$

$$P^{--}_{(1,2),3} = \cos(\beta_2)\,U3 - \cos(\beta_1)\sin(\beta_2)\,U2 + \sin(\beta_1)\sin(\beta_2)\,U1$$

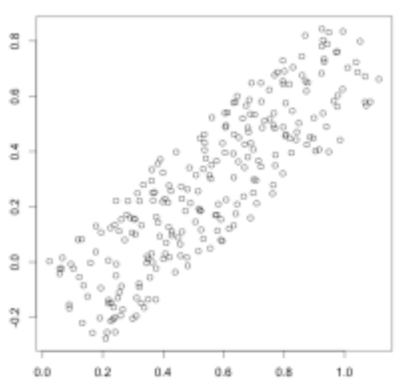
$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos\alpha_3 & \sin\alpha_3 \\ 0 & 0 & -\sin\beta_3 & \cos\beta_3 \end{bmatrix}\begin{bmatrix} P^{+}_{(1,2)} \\ P^{-+}_{(1,2),3} \\ P^{--}_{(1,2),3} \\ U_4 \end{bmatrix} = \begin{bmatrix} P^{+}_{(1,2)} \\ P^{-+}_{(1,2),3} \\ P^{--+}_{[(1,2),3],4} \\ P^{---}_{[(1,2),3],4} \end{bmatrix}$$

Procedure

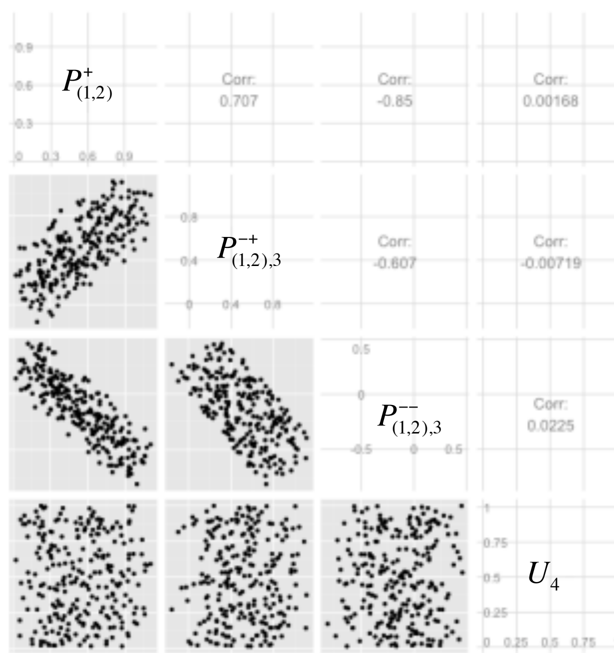
Independence between U_1 *and* U_2



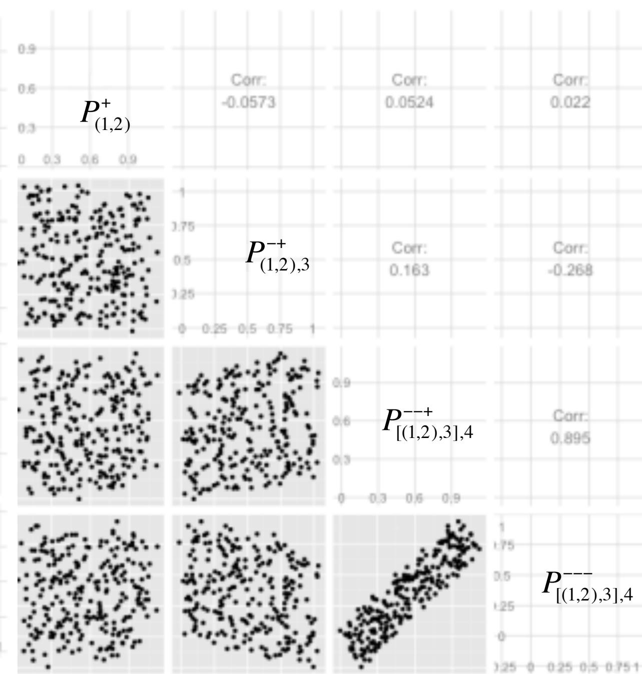
$P^{+}_{(1,2)}$ *vs* $P^{1}_{(1,2)}$ *with* $\alpha_1 = 80^\circ, \beta_1 = 20^\circ$



$\alpha_1 = 81^\circ, \beta_1 = 18^\circ, \alpha_2 = 33^\circ, \beta_2 = 73^\circ$



$\alpha_1 = 10^\circ, \beta_1 = 25^\circ, \alpha_2 = 85^\circ, \beta_2 = 15^\circ, \alpha_3 = 81^\circ, \beta_3 = 17^\circ$

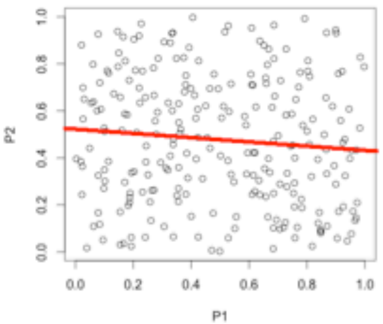


Alpha value	Beta Value	Correlation
75	25	-0.313
25	75	0.213
30	60	0.988
35	55	-0.915
35	65	0.987
35	60	0.0228
35	58	0.854
35	62	-0.951
62	35	0.962

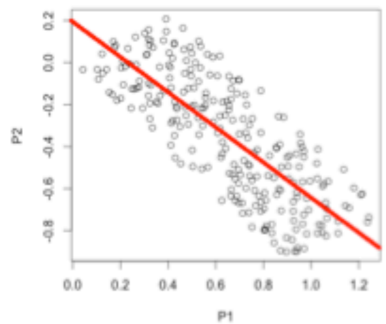
(75,25) -		(30,60) +	(25,75) +	(10,10) N	(20,40) -	(62,35) +
	P1 Q2	No -0.0639	Bad N -0.309	Low P 0.255	Low N -0.103	Low N -0.208
	P1 Q3	Low N -0.105	No -0.0965	Low N -0.171	Low P 0.22	Low N -0.139
	Q2 Q3	Very High P 0.991	Low P 0.147	No 0.0432	High N -0.904	Very High P 0.965
(20,80) +						
	P1 Q2	Low P 0.125	High P 0.941	Low N -0.167	Low P 0.289	Some P 0.648
	P1 Q3	Low P 0.161	Low P 0.269	Low P 0.212	Some N -0.649	Okay P 0.444
	Q2 Q3	Very High P 0.988	Low P 0.147	No 0.00566	High N -0.904	Very High P 0.965

First	**Second**		(25,75) +	(5,10) +	(7,10) -	(35,65) +	(72,25) N/+	(35, 55) -
(75,25) -	(10,10) -	P1 Q2	Low P 0.255	low P 0.255	low P 0.255	Low P 0.255	Low P 0.255	Low P 0.255
		P1 R3	Low N -0.175	No -0.0885	Low N -0.103	Low P 0.138	Low P 0.175	Low P 0.138
		P1 R4	No -0.0292	Low N -0.128	Low N -0.128	Low P 0.12	No 0.0181	Low N -0.17
		Q2 R3	No 0.0441	No 0.0219	No 0.0265	No -0.0351	No -0.0442	No -0.0351
		Q2 R4	No 0.00782	No 0.0319	No 0.0319	No -0.0305	No -0.0041	No 0.043
		R3 R4	Low P 0.259	Very High P 0.959	Low N -0.124	Very High P 0.988	Low P 0.13	High N -0.914

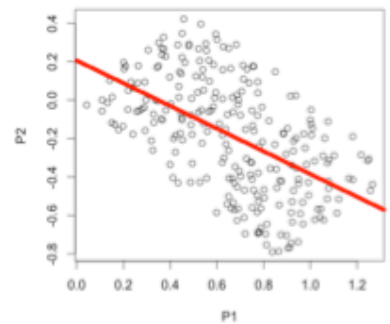
Procedure



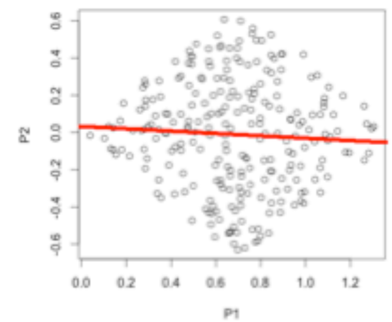
P(0 deg, 0 deg)
Correlation=-0.09653516



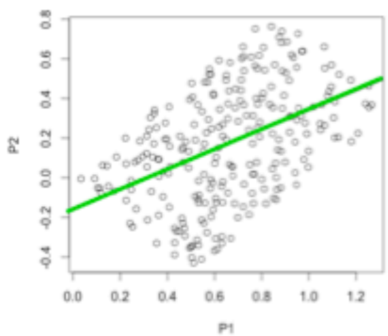
P(25 deg, 75 deg)
Correlation=-0.783814



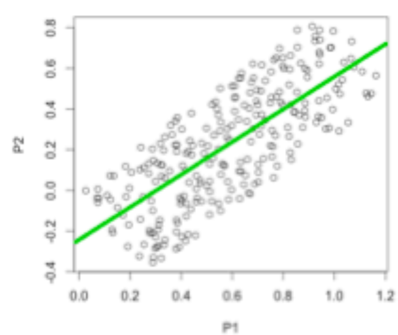
P(30 deg, 60 deg)
Correlation=-0.5439036



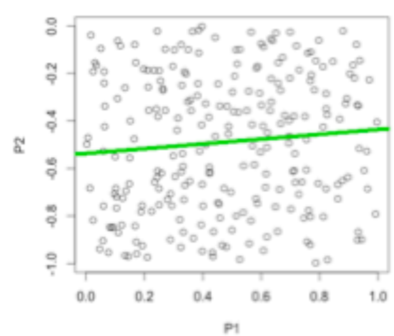
P(45 deg, 45 deg)
Correlation=-0.05804009



P(60 deg, 30 deg)
Correlation=0.457075



P(75 deg, 25 deg)
Correlation=0.7507463



P(90 deg, 90 deg)
Correlation=0.09653516

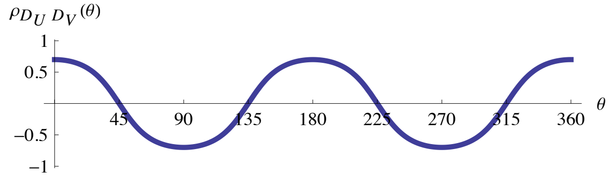
Process

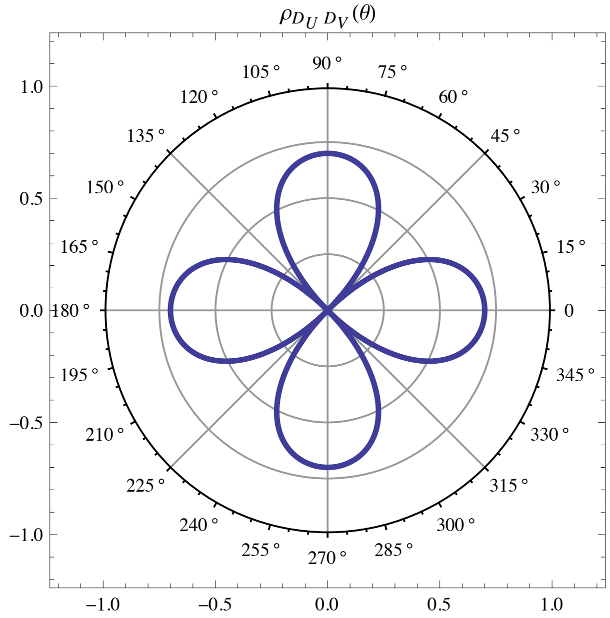
- Find exact correlations
- We know correlations between U_i's are 0

$$\mathrm{Corr}(P^{+}_{(1,2)}, P^{-}_{(1,2)}) = Sin(\alpha_1 - \beta_1)$$
$$\mathrm{Corr}(P^{+}_{(1,2)}, P^{-+}_{(1,2),3}) = Cos(\alpha_2)Sin(\alpha_1 - \beta_1)$$
$$\mathrm{Corr}(P^{+}_{(1,2)}, P^{--}_{(1,2),3}) = -Sin(\alpha_1 - \beta_1)Sin(\beta_2)$$
$$\mathrm{Corr}(P^{-+}_{(1,2),3}, P^{--}_{(1,2),3}) = Sin(\alpha_2 - \beta_2)$$
$$\mathrm{Corr}(P^{+}_{(1,2)}, P^{--+}_{[(1,2),3],4}) = -Cos(\alpha_3)Sin(\alpha_1 - \beta_1)Sin(\beta_2)$$
$$\mathrm{Corr}(P^{+}_{(1,2)}, P^{---}_{[(1,2),3],4}) = Sin(\beta_3)Sin(\alpha_1 - \beta_1)Sin(\beta_2)$$
$$\mathrm{Corr}(P^{-+}_{(1,2),3}, P^{--+}_{[(1,2),3],4}) = Cos(\alpha_3)Sin(\alpha_2 - \beta_2)$$
$$\mathrm{Corr}(P^{-+}_{(1,2),3}, P^{---}_{[(1,2),3],4}) = -Sin(\beta_3)Sin(\alpha_2 - \beta_2)$$
$$\mathrm{Corr}(P^{--+}_{[(1,2),3],4}, P^{---}_{[(1,2),3],4}) = Sin(\alpha_3 - \beta_3)$$

Result

- 2012
- One angle





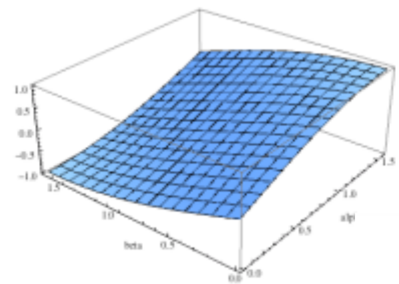
Result

Theorem 1. Suppose that (U_i, U_j) are independent uniform random variables on the interval (0,1).

Define $P_1 = \cos(\alpha)U_i + \sin(\alpha)U_j$ and $P_2 = -\sin(\beta)U_i + \cos(\beta)U_j$. Then,

(1) $$Cov(P(\alpha,\beta)) = \begin{bmatrix} 1/12 & (1/12)\sin(\alpha-\beta) \\ (1/12)\sin(\alpha-\beta) & 1/12 \end{bmatrix}$$

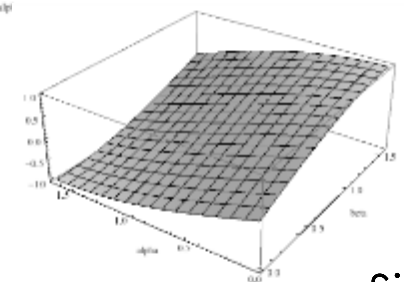
$$Cor(P(\alpha,\beta)) = \begin{bmatrix} 1 & \sin(\alpha-\beta) \\ \sin(\alpha-\beta) & 1 \end{bmatrix}$$



Sin(β - α)

(2) Cor(α , β)=-Cor(β , α)

(3) If $\alpha = \beta = \theta$, then (P_1, P_2) are independent.



Sin(α - β)

Result

Theorem 2. Suppose that (U_1,U_2,U_3,U_4) are independent uniform random variables on the interval (0,1). Define

$$\begin{bmatrix} \cos\alpha_1 & \sin\alpha_1 & 0 & 0 \\ -\sin\beta_1 & \cos\beta_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \end{bmatrix} = \begin{bmatrix} P^{+}_{(1,2)} \\ P^{-}_{(1,2)} \\ U_3 \\ U_4 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\alpha_2 & \sin\alpha_2 & 0 \\ 0 & -\sin\beta_2 & \cos\beta_2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} P^{+}_{(1,2)} \\ P^{-}_{(1,2)} \\ U_3 \\ U_4 \end{bmatrix} = \begin{bmatrix} P^{+}_{(1,2)} \\ P^{-+}_{(1,2),3} \\ P^{--}_{(1,2),3} \\ U_4 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos\alpha_3 & \sin\alpha_3 \\ 0 & 0 & -\sin\beta_3 & \cos\beta_3 \end{bmatrix} \begin{bmatrix} P^{+}_{(1,2)} \\ P^{-+}_{(1,2),3} \\ P^{--}_{(1,2),3} \\ U_4 \end{bmatrix} = \begin{bmatrix} P^{+}_{(1,2)} \\ P^{-+}_{(1,2),3} \\ P^{--+}_{[(1,2),3],4} \\ P^{---}_{[(1,2),3],4} \end{bmatrix}$$

Then (1) $Cov\left(P^{+}_{(1,2)}, P^{-+}_{(1,2),3}\right) = \begin{bmatrix} 1/12 & (1/12)\cos(\alpha_2)\sin(\alpha_1-\beta_1) \\ (1/12)\cos(\alpha_2)\sin(\alpha_1-\beta_1) & 1/12 \end{bmatrix}$

$Cor\left(P^{+}_{(1,2)}, P^{-+}_{(1,2),3}\right) = \begin{bmatrix} 1 & \cos(\alpha_2)\sin(\alpha_1-\beta_1) \\ \cos(\alpha_2)\sin(\alpha_1-\beta_1) & 1 \end{bmatrix}$

(2) $Cov\left(P^{+}_{(1,2)}, P^{--+}_{[(1,2),3],4}\right) = \begin{bmatrix} 1/12 & -(1/12)\cos(\alpha_3)\sin(\alpha_1-\beta_1)\sin(\beta_2) \\ -(1/12)\cos(\alpha_3)\sin(\alpha_1-\beta_1)\sin(\beta_2) & 1/12 \end{bmatrix}$

$Cor\left(P^{+}_{(1,2)}, P^{--+}_{[(1,2),3],4}\right) = \begin{bmatrix} 1 & -\cos(\alpha_3)\sin(\alpha_1-\beta_1)\sin(\beta_2) \\ -\cos(\alpha_3)\sin(\alpha_1-\beta_1)\sin(\beta_2) & 1 \end{bmatrix}$

(3) $Cov\left(P^{+}_{(1,2)}, P^{---}_{[(1,2),3],4}\right) = \begin{bmatrix} 1/12 & (1/12)\sin(\beta_3)\sin(\alpha_1-\beta_1)\sin(\beta_2) \\ (1/12)\sin(\beta_3)\sin(\alpha_1-\beta_1)\sin(\beta_2) & 1/12 \end{bmatrix}$

$Cor\left(P^{+}_{(1,2)}, P^{---}_{[(1,2),3],4}\right) = \begin{bmatrix} 1 & \sin(\beta_3)\sin(\alpha_1-\beta_1)\sin(\beta_2) \\ \sin(\beta_3)\sin(\alpha_1-\beta_1)\sin(\beta_2) & 1 \end{bmatrix}$

(4) $Cov\left(P^{-+}_{(1,2),3}, P^{--+}_{[(1,2),3],4}\right) = \begin{bmatrix} 1/12 & (1/12)\cos(\alpha_3)\sin(\alpha_2-\beta_2) \\ (1/12)\cos(\alpha_3)\sin(\alpha_2-\beta_2) & 1/12 \end{bmatrix}$

$Cor\left(P^{-+}_{(1,2),3}, P^{--+}_{[(1,2),3],4}\right) = \begin{bmatrix} 1 & \cos(\alpha_3)\sin(\alpha_2-\beta_2) \\ \cos(\alpha_3)\sin(\alpha_2-\beta_2) & 1 \end{bmatrix}$

(5) $Cov\left(P^{-+}_{(1,2),3}, P^{---}_{[(1,2),3],4}\right) = \begin{bmatrix} 1/12 & -(1/12)\sin(\beta_3)\sin(\alpha_2-\beta_2) \\ -(1/12)\sin(\beta_3)\sin(\alpha_2-\beta_2) & 1/12 \end{bmatrix}$

$Cor\left(P^{-+}_{(1,2),3}, P^{---}_{[(1,2),3],4}\right) = \begin{bmatrix} 1 & -\sin(\beta_3)\sin(\alpha_2-\beta_2) \\ -\sin(\beta_3)\sin(\alpha_2-\beta_2) & 1 \end{bmatrix}$

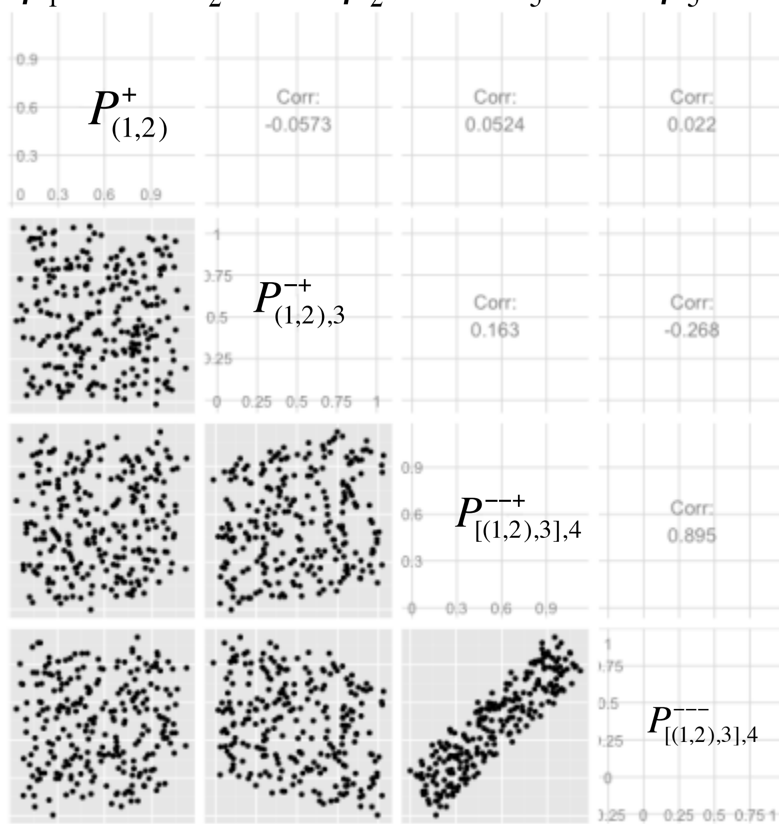
(6) $Cov\left(P^{--+}_{[(1,2),3],4}, P^{---}_{[(1,2),3],4}\right) = \begin{bmatrix} 1/12 & (1/12)\sin(\alpha_3-\beta_3) \\ (1/12)\sin(\alpha_3-\beta_3) & 1/12 \end{bmatrix}$

$Cor\left(P^{--+}_{[(1,2),3],4}, P^{---}_{[(1,2),3],4}\right) = \begin{bmatrix} 1 & \sin(\alpha_3-\beta_3) \\ \sin(\alpha_3-\beta_3) & 1 \end{bmatrix}$

Result

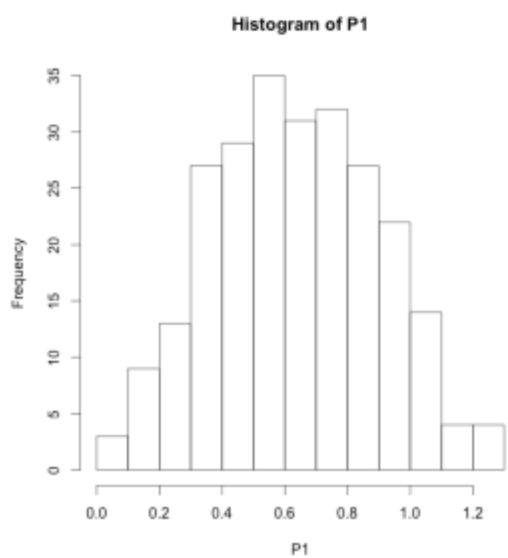
Then, a matrix plot of $P^{+}_{(1,2)}, P^{-+}_{(1,2),3}, P^{--+}_{[(1,2),3],4}, P^{---}_{[(1,2),3],4}$ where

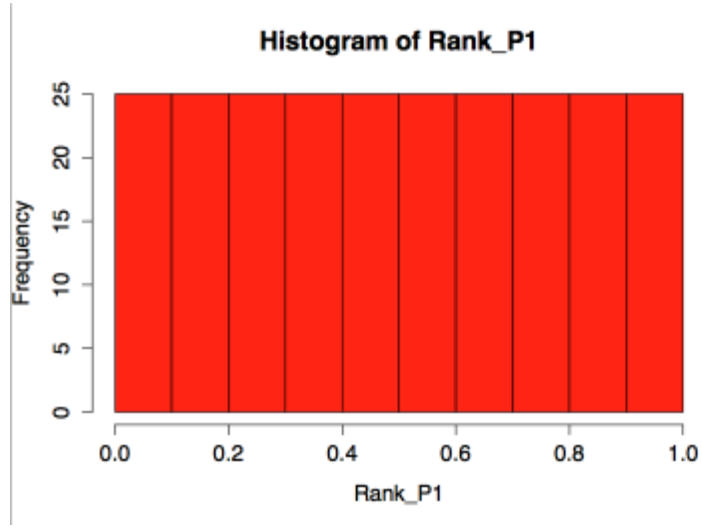
$$\alpha_1 = 10^\circ, \beta_1 = 25^\circ, \alpha_2 = 85^\circ, \beta_2 = 15^\circ, \alpha_3 = 81^\circ, \beta_3 = 17^\circ$$



Result

- Not uniform anymore
- Rank





Simulation

- Motion Chart

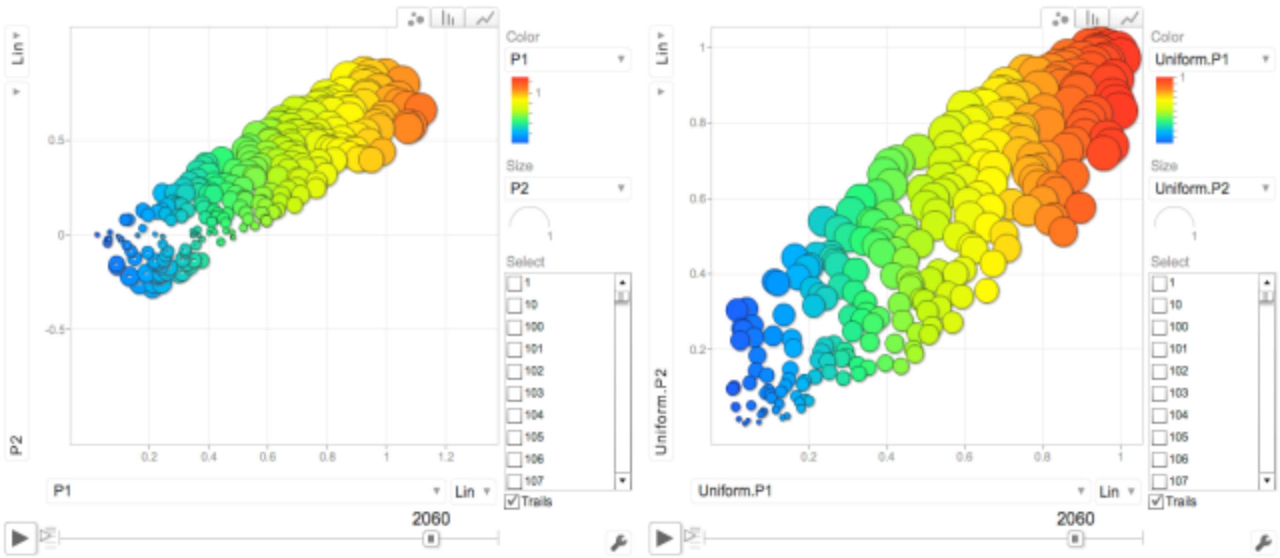


Figure. Plots of projections and their uniform transformations as a function of α-β.

Note: *Scale shows 2000+(α-β)*

Future Research

- Explore properties up to ‘n’ variables
- Find a copula with directional dependence property
- Apply these methods to a data set

References

[1] Nelsen, R. B., *An Introduction to copulas* (2nd edn.), New York: Springer, 2006

[2] Sungur, E.A., Orth, J.M., *Constructing a New Class of Copulas with Directional Dependence Property by Using Oblique Jacobi Transformations*, 2012

[3] Sungur, E.A., Orth, J.M., *Understanding Directional Dependence through Angular Correlations*, 2011