Directional Dependence using Copulas

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Abstract

Directional dependence and copulas are relatively new topics in the statistics discipline. Directional dependence relies on how the data is looked at; understanding it using copulas can be rather puzzling, but will be helpful amongst a variety of applications. In this research, we further develop the ideas of spatial statistics and Sungur's previous research on this approach to look at directional dependence using two direction parameters expressed as angles. This approach will help us understand the dependence structure and help create a model using copulas which will allow us, in the future, to draw conclusions of dependence in informative data. A general format of rotations and projections of different angles give us different viewpoints of the dependence structures in two, three, four, or more random variables.

Introduction

Direction of dependence simply utilizes positive or negative correlations. Directional dependence goes deeper and is not merely limited to positive and negative correlation, but considers causation and pattern changes. To try and emphasize directional dependence, we explore different angles in the angular parameters, alpha and beta, to view the data and investigate properties through rotations and projections.

Theorem 1*. Suppose that (U_i,U_j) are independent uniform random variables on the interval (0,1). Define $P_1=\cos(\alpha)U_i+\sin(\alpha)U_j$ and $P_2=-\sin(\beta)U_i+\cos(\beta)U_j$. Then,*

(1)

$$Cov(P_1,P_2)=\begin{bmatrix} 1/12 & (1/12)\sin(\alpha-\beta) \\ (1/12)\sin(\alpha-\beta) & 1/12 \end{bmatrix}$$

$$Cor(P_1,P_2)=\begin{bmatrix} 1 & \sin(\alpha-\beta) \\ \sin(\alpha-\beta) & 1 \end{bmatrix}$$

(2) $Cor(\alpha,\beta)=-Cor(\beta,\alpha)$

(3) If $\alpha=\beta=\theta$, then (P_1,P_2) are independent.

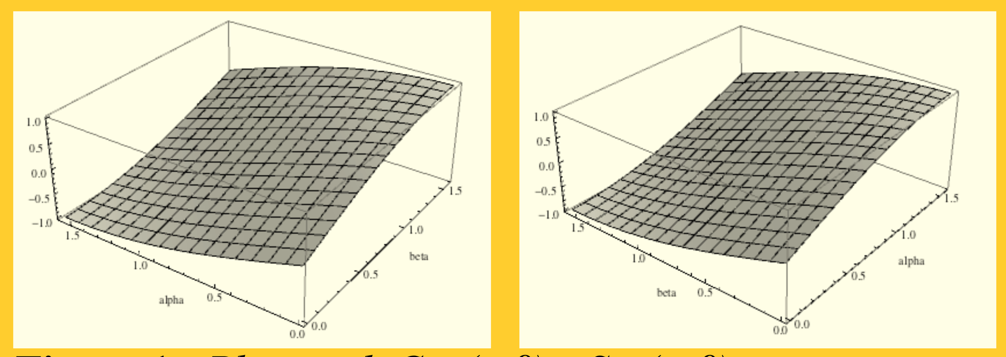

Figure 1. *Plots with $Cor(\alpha,\beta)=\sin(\alpha-\beta)$, $Cor(\alpha,\beta)=\sin(\beta-\alpha)$ respectively.*

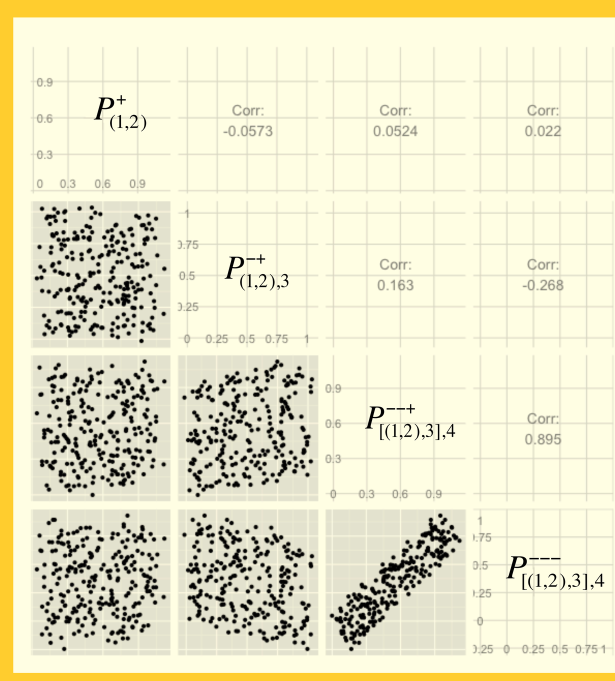

Figure 2. *Matrix plot of $P^{+}_{(1,2)}, P^{-+}_{(1,2),3}, P^{--+}_{[(1,2),3],4}, P^{---}_{[(1,2),3],4}$ with angles* $\alpha_1=10^\circ, \beta_1=25^\circ, \alpha_2=85^\circ, \beta_2=15^\circ, \alpha_3=81^\circ, \beta_3=17^\circ$

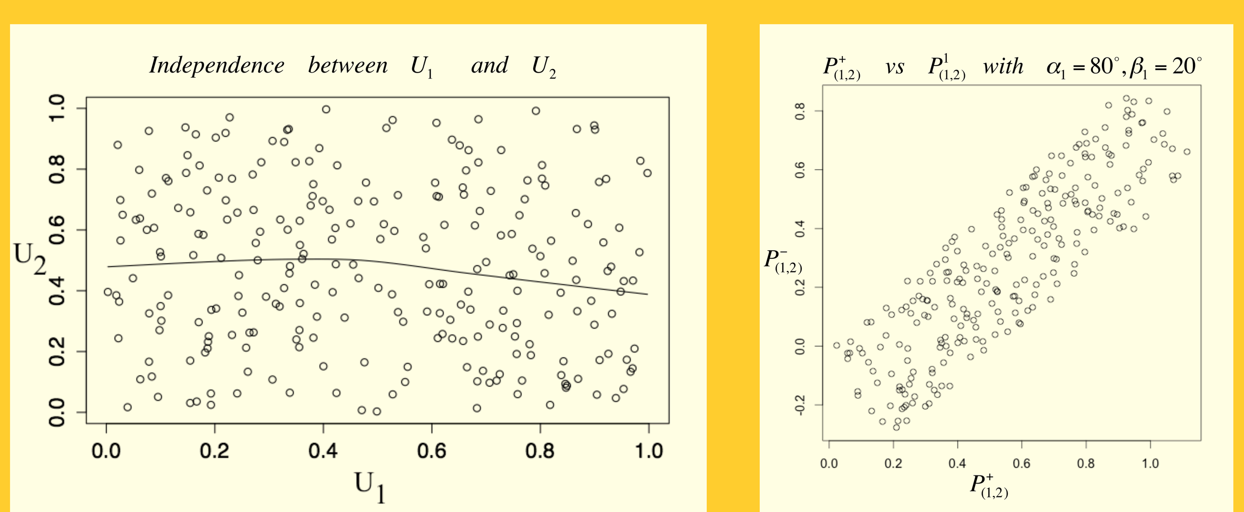

Figure 3. *Plots of projections showing their dependence structure before and after rotation.*

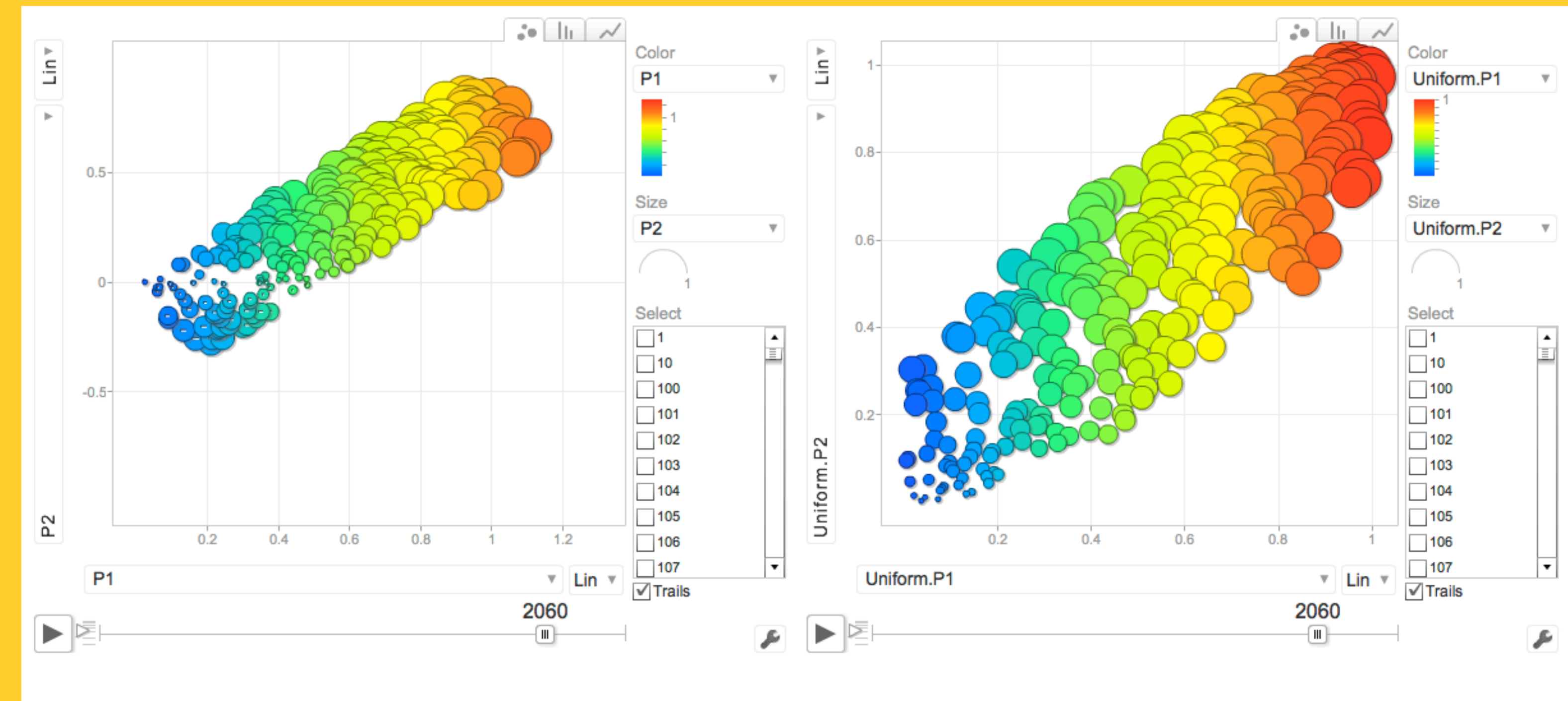

Figure 4. Plots of projections and their uniform transformations as a function of α-β.

Note: *Scale shows 2000+(α-β)*

Theorem 2*. Suppose that (U_1,U_2,U_3,U_4) are independent uniform random variables on the interval (0,1). Define*

$$\begin{bmatrix} \cos\alpha_1 & \sin\alpha_1 & 0 & 0 \\ -\sin\beta_1 & \cos\beta_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \end{bmatrix}=\begin{bmatrix} P^{+}_{(1,2)} \\ P^{-}_{(1,2)} \\ U_3 \\ U_4 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\alpha_2 & \sin\alpha_2 & 0 \\ 0 & -\sin\beta_2 & \cos\beta_2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} P^{+}_{(1,2)} \\ P^{-}_{(1,2)} \\ U_3 \\ U_4 \end{bmatrix}=\begin{bmatrix} P^{+}_{(1,2)} \\ P^{-+}_{(1,2),3} \\ P^{--}_{(1,2),3} \\ U_4 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos\alpha_3 & \sin\alpha_3 \\ 0 & 0 & -\sin\beta_3 & \cos\beta_3 \end{bmatrix}\begin{bmatrix} P^{+}_{(1,2)} \\ P^{-+}_{(1,2),3} \\ P^{--}_{(1,2),3} \\ U_4 \end{bmatrix}=\begin{bmatrix} P^{+}_{(1,2)} \\ P^{-+}_{(1,2),3} \\ P^{--+}_{[(1,2),3],4} \\ P^{---}_{[(1,2),3],4} \end{bmatrix}$$

Then *(1)*

$$Cov\left(P^{+}_{(1,2)},P^{-+}_{(1,2),3}\right)=\begin{bmatrix} 1/12 & (1/12)\cos(\alpha_2)\sin(\alpha_1-\beta_1) \\ (1/12)\cos(\alpha_2)\sin(\alpha_1-\beta_1) & 1/12 \end{bmatrix}$$

$$Cor\left(P^{+}_{(1,2)},P^{-+}_{(1,2),3}\right)=\begin{bmatrix} 1 & \cos(\alpha_2)\sin(\alpha_1-\beta_1) \\ \cos(\alpha_2)\sin(\alpha_1-\beta_1) & 1 \end{bmatrix}$$

(2)

$$Cov\left(P^{+}_{(1,2)},P^{--+}_{[(1,2),3],4}\right)=\begin{bmatrix} 1/12 & -(1/12)\cos(\alpha_3)\sin(\alpha_1-\beta_1)\sin(\beta_2) \\ -(1/12)\cos(\alpha_3)\sin(\alpha_1-\beta_1)\sin(\beta_2) & 1/12 \end{bmatrix}$$

$$Cor\left(P^{+}_{(1,2)},P^{--+}_{[(1,2),3],4}\right)=\begin{bmatrix} 1 & -\cos(\alpha_3)\sin(\alpha_1-\beta_1)\sin(\beta_2) \\ -\cos(\alpha_3)\sin(\alpha_1-\beta_1)\sin(\beta_2) & 1 \end{bmatrix}$$

(3)

$$Cov\left(P^{+}_{(1,2)},P^{---}_{[(1,2),3],4}\right)=\begin{bmatrix} 1/12 & (1/12)\sin(\beta_3)\sin(\alpha_1-\beta_1)\sin(\beta_2) \\ (1/12)\sin(\beta_3)\sin(\alpha_1-\beta_1)\sin(\beta_2) & 1/12 \end{bmatrix}$$

$$Cor\left(P^{+}_{(1,2)},P^{---}_{[(1,2),3],4}\right)=\begin{bmatrix} 1 & \sin(\beta_3)\sin(\alpha_1-\beta_1)\sin(\beta_2) \\ \sin(\beta_3)\sin(\alpha_1-\beta_1)\sin(\beta_2) & 1 \end{bmatrix}$$

(4)

$$Cov\left(P^{-+}_{(1,2),3},P^{--+}_{[(1,2),3],4}\right)=\begin{bmatrix} 1/12 & (1/12)\cos(\alpha_3)\sin(\alpha_2-\beta_2) \\ (1/12)\cos(\alpha_3)\sin(\alpha_2-\beta_2) & 1/12 \end{bmatrix}$$

$$Cor\left(P^{-+}_{(1,2),3},P^{--+}_{[(1,2),3],4}\right)=\begin{bmatrix} 1 & \cos(\alpha_3)\sin(\alpha_2-\beta_2) \\ \cos(\alpha_3)\sin(\alpha_2-\beta_2) & 1 \end{bmatrix}$$

(5)

$$Cov\left(P^{-+}_{(1,2),3},P^{---}_{[(1,2),3],4}\right)=\begin{bmatrix} 1/12 & -(1/12)\sin(\beta_3)\sin(\alpha_2-\beta_2) \\ -(1/12)\sin(\beta_3)\sin(\alpha_2-\beta_2) & 1/12 \end{bmatrix}$$

$$Cor\left(P^{-+}_{(1,2),3},P^{---}_{[(1,2),3],4}\right)=\begin{bmatrix} 1 & -\sin(\beta_3)\sin(\alpha_2-\beta_2) \\ -\sin(\beta_3)\sin(\alpha_2-\beta_2) & 1 \end{bmatrix}$$

(6)

$$Cov\left(P^{--+}_{[(1,2),3],4},P^{---}_{[(1,2),3],4}\right)=\begin{bmatrix} 1/12 & (1/12)\sin(\alpha_3-\beta_3) \\ (1/12)\sin(\alpha_3-\beta_3) & 1/12 \end{bmatrix}$$

$$Cor\left(P^{--+}_{[(1,2),3],4},P^{---}_{[(1,2),3],4}\right)=\begin{bmatrix} 1 & \sin(\alpha_3-\beta_3) \\ \sin(\alpha_3-\beta_3) & 1 \end{bmatrix}$$

Future Research

In the next stages, we are aiming to further explore the properties of such structures with up to 'n' variables, find a copula with directional dependence property, and apply these methods to a data set to try understanding the dependence structure.

Objectives

- Understanding of spatial statistics and copulas
- Determine how to develop and the developing of new models
- Future application to data

Research Procedure

- Begin with four uniform sets of 250 randomly generated numbers ranging from 0 to 1 for simulation. Sets are labeled U_1, U_2, U_3, U_4.
- Understand the format and simulate projections and rotations using RStudio software.
- Investigate the effects of different angles through a variety of graphical display and correlation estimation.
- Find exact correlations for all projections in rotations
- Transform the projections back to a uniform distribution through rank method.
- Compare original projection behavior to transformed behavior through a motion chart

Conclusion

In this research we have introduced a method to understand directional dependence using the ideas of spatial statistics, copulas, and previous research. By generalizing the idea of an angular parameter into two parameters, we have created a snapshot of a model and the hidden dependence structure. We have also uncovered new properties and a baseboard to find copulas with directional dependence property.

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