ON MODELING DIRECTIONAL DEPENDENCE BY USING COPULAS

Engin A. Sungur[1]

Jessica M. Orth[2]

ABSTRACT

Understanding and modeling multivariate dependence structures depending upon the direction are challenging but an interest of theoretical and applied researchers. In this article, we introduce a way of looking at directional dependence by using a direction parameter, possibly random in Bayesian setting, expressed as an angle. This construction allows us to model and measure directional dependence in a meaningful way and leads to informative graphical displays. Our focus in this paper will be on the 3-dimensional case.

Keywords: Canonical correlation, Copulas, Directional dependence, Angular correlation, Spatial statistics

1. INTRODUCTION

[1] E. A. Sungur
Statistics Discipline, University of Minnesota, Morris
600 E. 4th Street, Morris, MN 56267, USA
email: sungurea@umn.edu

[2] J. M. Orth
Statistics Discipline, University of Minnesota, Morris
600 E. 4th Street, Morris, MN 56267, USA
email: orthx055@umn.edu

Directional dependence is a concept that can be defined in various ways depending upon the researcher's interest. It needs to be clarified at the beginning that directional dependence is different than the direction of dependence. Direction of dependence implies changes in the behavior of underlying variables relative to the changes on other set of variables. As it is discussed by Nelsen and Úbeda-Flores (2011), in bivariate setting, if large (small) values of the random variables tend to occur together then the direction of dependence is positive; and if the large values of one variable tend to occur with small values of the other then the direction of dependence is negative. On the other hand directional dependence means changes in the joint behavior of the underlying variables based on the order that they have been introduced or the "way" that it has been looked at. Therefore, in directional dependence not only the dichotomy of positive and negative dependence but also any change in the pattern including the magnitude plays an important role.

In most of these cases in application the objective is to be able to set up causal relationship which can only be set through experiments. In this case the problem can be stated as being able to distinguish between the exploratory and response variables. Dodge and Rousson (2000), Muddapur (2005), and Sungur (2005) studied directional dependence in regression setting. In the first two, they suggested an approach to decide about the direction of regression line. Their approach is based on the use of the coefficients of skewness of the two random variables. It has been shown that under the assumption that the error term is symmetric the cube of the Pearson's correlation can be calculated as the ratio of these two skewness measures. Dodge and Rousson (2000) claimed that the deviation from symmetry can be used to assess the directional dependence and the response variable will always have less skewness than the explanatory variable. Sungur (2005) argued that the conditional not the marginal behavior of the variables will determine the dependence structure and suggested the use of copula approach. In this case, existence of directional dependence in regression setting implies a different copula regression function in two directions and requires nonsymmetric copula.

Nelsen and Úbeda-Flores (2011) approach is basically on the direction of dependence and not on directional dependence as we have defined. They propose measures to detect dependence in multivariate distributions that cannot be detected by some well known measures of association. Their approach is limited to the eight "directions" determined by a vector $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \alpha_3)$, where $\alpha_i \epsilon \{-1,1\}$. Briefly, they consider the random triplet (U, V, W) with uniform marginals on the unit interval with joint distribution function (copula) C, define the eight possible transformed random vectors of the form $\left((-1)^i U, (-1)^j V, (-1)^k W\right)$, $i, j, k = 1{,}2$, and find the generalized Pearson's correlation of them. If we represent the copula of each transformed vector by C^{ijk}, then the generalized Pearson's correlation is $\rho_3^{C^{ijk}} = 8\int_0^1\int_0^1\int_0^1 \left[C^{ijk}(u, v, w) - uvw\right] dudvdw$. Note that C^{ijk} is not the joint distribution of the transformed variables but the corresponding copula. It is interesting to note that even if they use the joint distribution of the transformed vector, the results are the same as using the copulas.

Sungur and Celebioglu (2011) discussed generating class of copulas with directional dependence property by introducing direction covariate and set up the connection between directed graphs. In this paper, we will argue that dependence structures change depending on the direction chosen. Therefore, changing our "orientation" may lead to a different view on how the variables are related. In simple terms this orientation can be associated with a direction parameter such as an angle, possibly with a specified prior distribution.

Our approach was based on a set up commonly used in spatial statistics. Simon(1997) suggested a way of replacing spatial correlations by ordinary correlations by projecting the two dimensional coordinates of a map onto a single axis making angle θ with the horizontal (east-west) axis. By doing so he developed angular correlation plots that were meaningful and informative. Simon considered observed values of a random variable X at n locations that each could be identified with a point (s, t) in the coordinate system. Consideration of an angle θ, such that $\theta = 0^o$ ($\theta = 90^o$) showing the direction of east (north), gave him an opportunity to plot dependence measure as a function of an angle. He used θ as an index, defined $P(\theta) = scos(\theta) + tsin(\theta)$, and worked on the Pearson's correlation between X and $P(\theta)$. The $P(\theta)$ could be interpreted as the X's "coordinate" along the axis pointing in the direction θ, i.e., the dot product of the original point by a normalized length vector $\big(cos(\theta), sin(\theta)\big)$. In his paper, the sample version of this measure was presented. In Section 2 we will formalize our approach by introducing the concept of random coordinates and direction parameter in an angle form. Also the distribution of the appropriately defined random pair, one being the rotation/projection will be discussed. Section 3 will present a new measure of directional dependence and its properties, and Section 4 will provide an application of our approach to Minnesota county data.

2. BASIC DEFINITIONS AND SET-UP

Let (X, Y, Z) be a 3-dimensional random vector with joint distribution function F, and univariate marginal distributions $F_X, F_Y,$ and F_Z. Since dependence between random variables is not related with the marginal behavior of variables, we will work with copulas.

Definition 1: A 3-dimensional copula is a function C with the following properties:

a. For every (u, v, w) in $\mathbf{I^3} = [0,1]\times[0,1]\times[0,1]$,

   $C(u, v, w) = 0$ if at least one coordinate is 0, and

   if all coordinates are 1 except one, say u, then $C(u, v, w) = u$;

b. For every (u_1, v_1, w_1) and (u_2, v_2, w_2) in $\mathbf{I^3}$ such that $(u_1, v_1, w_1) \leq (u_2, v_2, w_2)$,

$$C(u_2, v_2, w_2) - C(u_2, v_2, w_1) - C(u_2, v_1, w_2) - C(u_1, v_2, w_2)$$
$$+C(u_2, v_1, w_1) + C(u_1, v_2, w_1) + C(u_1, v_1, w_2) - C(u_1, v_1, w_1) \geq 0.$$

Let (U, V, W) be a 3-dimensional random vector with uniform marginal distributions over the interval [0, 1]. Also, let $C(u, v, w) = P\{U \leq u, V \leq v, W \leq w\}$ be the joint distribution of (U, V, W).Then the function C is a copula, see Sklar (1959), and Nelsen (2006). Sklar (1959) proved that any continuous multivariate distribution function can be written as a function of its univariate marginal distribution functions via copula. That is for any 3-dimensional distribution function there exists an 3-copula *C* such that $F(x, y, z) = C\big(F_X(x), F_Y(y), F_Z(z)\big)$ for all (x, y, z) in extended 3-space $[-\infty, \infty]\times[-\infty, \infty]\times[-\infty, \infty]$. Furthermore, if the univariate marginal are continuous then *C* is unique.

In this paper, we will consider transformed random vector $\big(U = F_X(X), V = F_Y(Y), W = F_Z(Z)\big)$. Similar to the basic set up used in spatial statistics, we will view one of the random variables, say *U*, as a random quantity located at a random position *(V, W)* in the two-dimensional Euclidean space. Consider an angle θ, where $\theta = 0^o$ is in the direction of east, i.e. variable *V*. We will increase θ counterclockwise, so that $\theta = 90^o$ corresponds to north, i.e. variable *W*. We will regard θ as the direction parameter. Now, let us define

$$P_{VW}(\theta) = cos(\theta)V + sin(\theta)W,$$

which can be interpreted as the random variable *U*'s "coordinate" along the axis pointing in the direction θ in the Euclidean space defined by *V* and *W*. In other words it is the dot product of the original point/location by a normalized length vector $\big(cos(\theta), sin(\theta)\big)$.

Our main interest will be the distributional properties of $\big(U, P_{VW}(\theta)\big)$ including the Pearson's correlation. Let $C_{U,V|W}$ be the conditional copula of (U, V) given $W = w$, i.e.,

$$C_{U,V|W}(u, v|w) = \frac{\partial C(u, v, w)}{\partial w}$$

The distribution function of $\big(U, P_{VW}(\theta)\big)$ is

$$F_{UP_{VW}(\theta)}(u, p) = P\{U \leq u, P_{VW}(\theta) \leq p\} = \int_0^1 P\{U \leq u, P_{VW}(\theta) \leq p|W = w\}dw.$$

Depending upon the angle therefore the position of Uin the four quadrants we end up with two cases.

Case 1. The East-North and West-North quadrants i.e. $\theta \in [0°, 90°] \cup [90°, 180°]$

$P\{U \leq u, P_{VW}(\theta) \leq p|W = w\} = P\{U \leq u, Sin(\theta)V + Cos(\theta)W \leq p|W = w\} =$

$P\left\{U \leq u, V \leq \frac{p-Cos(\theta)w}{Sin(\theta)}|W = w\right\} = C_{U,V|W}\left(u, \frac{p-Cos(\theta)w}{Sin(\theta)}\right)$ for $0 \leq \frac{p-Cos(\theta)w}{Sin(\theta)} \leq 1$.

Case 2. The West-South and South-East quadrants, i.e. $\theta \in [180°, 270°] \cup [270°, 360°]$

$P\{U \leq u, P_{VW}(\theta) \leq p|W = w\} = u - C_{U,V|W}\left(u, \frac{p-Cos(\theta)w}{Sin(\theta)}\right)$ for $0 \leq \frac{p-Cos(\theta)w}{Sin(\theta)} \leq 1$.

In the next section, we will define and present results on a measure of dependence between U and $P_{VW}(\theta)$.

3. SOME MEASURES OF DIRECTIONAL DEPENDENCE

The Pearson's product moment correlation between $\left(U, P_{VW}(\theta)\right)$ can be expressed as a function of pairwise correlations of the random triplet (U, V, W). At this point we would like to clarify that arbitrary selection of pairwise correlations will not necessarily lead to meaningful values of the measure given below. These pairwise correlations need to be "compatible". In this paper we will only consider compatible pairwise correlations which are defined below.

Definition 2: The set of correlations (ρ_1, ρ_2, ρ_3) are said to be compatible if there exists a random vector (U, V, W) with 3-copula C such that $\rho_{UV} = \rho_1$, $\rho_{UW} = \rho_2,, and\ \rho_{VW} = \rho_3$.

Theorem 1. Let (U, V, W) be a random triplet with uniform marginal on the interval (0,1). Also let ρ_{UV}, ρ_{UW}, ρ_{VW} be the Pearson's correlation for (U, V), (U, W), and (V, W) respectively. Then

$$Cor\{U, P_{VW}(\theta)\} = \rho_{U|VW(\theta)} = \frac{\cos(\theta)\, \rho_{UV} + \sin(\theta)\rho_{UW}}{\sqrt{1 + \sin(2\theta)\rho_{VW}}}.$$

Proof.

First note that

$$E[U] = \frac{1}{2}, E[P_{VW}(\theta)] = E[cos(\theta)V + sin(\theta)W] = \frac{1}{2}\big(cos(\theta) + sin(\theta)\big),$$

$$Var[U] = \frac{1}{12}, Var[P_{VW}(\theta)] = \frac{1}{12}(1 + sin(2\theta)\rho_{VW}).$$

Also,

$$Cov\{U, P_{VW}(\theta)\} = E[UP_{VW}(\theta)] - E[U]E[P_{VW}(\theta)]$$

$$= cos(\theta)E[UV] + sin(\theta)E[UW] - \frac{1}{4}\big(cos(\theta) + sin(\theta)\big)$$

$$= cos(\theta)\frac{\rho_{UV} + 3}{12} + sin(\theta)\frac{\rho_{UW} + 3}{12} - \frac{1}{4}\big(cos(\theta) + sin(\theta)\big)$$

$$= \frac{1}{12}\Big[cos(\theta)(\rho_{UV} + 3) + sin(\theta)(\rho_{UW} + 3) - 3\big(cos(\theta) + sin(\theta)\big)\Big].$$

Therefore,

$$Cor\{U, P_{VW}(\theta)\} = \frac{Cov\{U, P_{VW}(\theta)\}}{\sqrt{Var[U]Var[P_{VW}(\theta)]}} = \frac{\cos(\theta)\, \rho_{UV} + \sin(\theta)\rho_{UW}}{\sqrt{1 + \sin(2\theta)\rho_{VW}}}.$$

Note that $\theta = 0^o$ will direct us to variable V (east), $\theta = 90^o$ to variable W (north), $\theta = 180^o$ to variable -V (west), and $\theta = 270^o$ to variable -W (south). East-north and west-south quadrants represent weighted averages of V, W and -V, -W, respectively. On the other hand, north-west and south-east quadrants will represent contrasts.

Corollary 1. $\rho_{U|VW(\theta+180°)} = -\rho_{U|VW(\theta)}$

Proof. The proof is straightforward through symmetry.

$$\rho_{U|VW(\theta+180°)} = \frac{\cos(\theta+180°)\,\rho_{UV} + \sin(\theta+180°)\rho_{UW}}{\sqrt{1+\sin(2(\theta+180°))\rho_{VW}}} =$$

$$\frac{-\cos(\theta)\,\rho_{UV} - \sin(\theta)\rho_{UW}}{\sqrt{1+\sin(2\theta)\rho_{VW}}} = -\frac{\cos(\theta)\,\rho_{UV} + \sin(\theta)\,\rho_{UW}}{\sqrt{1+\sin(2\theta)\,\rho_{VW}}} = -\rho_{U|VW(\theta)}$$

The relationship between these two equations is an inversion.

The plot of this measure as a function of θ will provide us with a valuable information on the direction of dependence. Fig. 1 provides angular correlation plots for the cases of perfect positive correlation between the variables (left), and $\rho_{UV} = 0.8$, $\rho_{UW} = 0.1$, $\rho_{VW} = 0.4$ (right). The solid and dashed sections of the curve indicate positive and negative correlation, respectively. From these plots, we can see how the dependence structure changes depending on the angle θ. In the plot on the left, the maximum correlation value of 1 is attained for $\theta \in [315°, 135°]$. In the plot on the right, the maximum correlation value of 0.8 is attained at $\theta = 335°$.

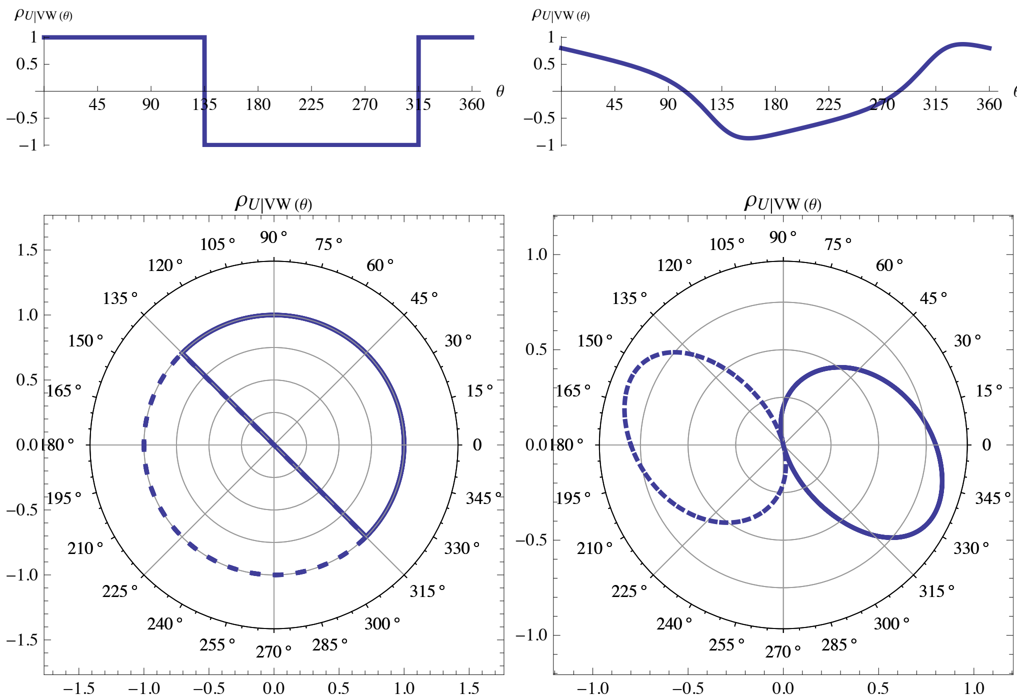


Fig. 1. Angular correlation plots of $\rho_{U|VW(\theta)}$ for the cases $\rho_{UV} = \rho_{UW} = \rho_{VW} = 1$ (left) and $\rho_{UV} = 0.8$ $\rho_{UW} = 0.1$, $\rho_{VW} = 0.4$ (right).

Now let us look at the direction at which the maximum absolute angular correlation will be attained. Our approach here is similar to the concept of canonical correlation that is discussed by Clark (1975) and Tishler and Lipovetsky (1996) and (2000). The canonical correlation analysis aims to create linear combinations based on two sets of random variables so that the maximum correlation between them can be obtained. The following result provides the directions/angles at which extreme angular correlations will occur as a function of pairwise correlations between the random triplet (U, V, W).

Theorem 2. Let *(U, V, W)* be a random triplet with uniform marginal on the interval (0,1). Also let ρ_{UV}, ρ_{UW}, ρ_{VW} be the Pearson's correlation for *(U, V)*, *(U, W)*, and *(V, W)*, respectively. Then, the extreme angular correlations will be attained at the following angles, given in radians:

$$\theta_{ext}^{1} = -ArcCos[-\frac{\rho_{UV} - \rho_{UW}\rho_{VW}}{\sqrt{\rho_{UV}^{2} + \rho_{UW}^{2} - 4\rho_{UV}\rho_{UW}\rho_{VW} + \rho_{UV}^{2}\rho_{VW}^{2} + \rho_{UW}^{2}\rho_{VW}^{2}}}]$$

$$\theta_{ext}^{2} = ArcCos[-\frac{\rho_{UV} - \rho_{UW}\rho_{VW}}{\sqrt{\rho_{UV}^{2} + \rho_{UW}^{2} - 4\rho_{UV}\rho_{UW}\rho_{VW} + \rho_{UV}^{2}\rho_{VW}^{2} + \rho_{UW}^{2}\rho_{VW}^{2}}}]$$

$$\theta_{ext}^{3} = -ArcCos[\frac{\rho_{UV} - \rho_{UW}\rho_{VW}}{\sqrt{\rho_{UV}^{2} + \rho_{UW}^{2} - 4\rho_{UV}\rho_{UW}\rho_{VW} + \rho_{UV}^{2}\rho_{VW}^{2} + \rho_{UW}^{2}\rho_{VW}^{2}}}]$$

$$\theta_{ext}^{4} = ArcCos[\frac{\rho_{UV} - \rho_{UW}\rho_{VW}}{\sqrt{\rho_{UV}^{2} + \rho_{UW}^{2} - 4\rho_{UV}\rho_{UW}\rho_{VW} + \rho_{UV}^{2}\rho_{VW}^{2} + \rho_{UW}^{2}\rho_{VW}^{2}}}]$$

Proof. The proof of this result is straightforward.

As an illustration, consider the case that $\rho_{UV} = 0.5$, $\rho_{UW} = 0.2$, $\rho_{VW} = -0.7$. The extreme angles are $\theta_{ext}^{1} = -2.43168$, $\theta_{ext}^{2} = 2.43168$, $\theta_{ext}^{3} = -0.709912$, and $\theta_{ext}^{4} = 0.709912$. In this case, the maximum correlation, 0.918225, is attained at $\theta = 0.709912\ (40.67^{o})$; the minimum correlation, -0.918225, is attained at $\theta = -2.43186\ (-139.33^{o})$. θ_{ext}^{1} and θ_{ext}^{2} yield the maximum correlation and θ_{ext}^{3} and θ_{ext}^{4} the minimum correlation.

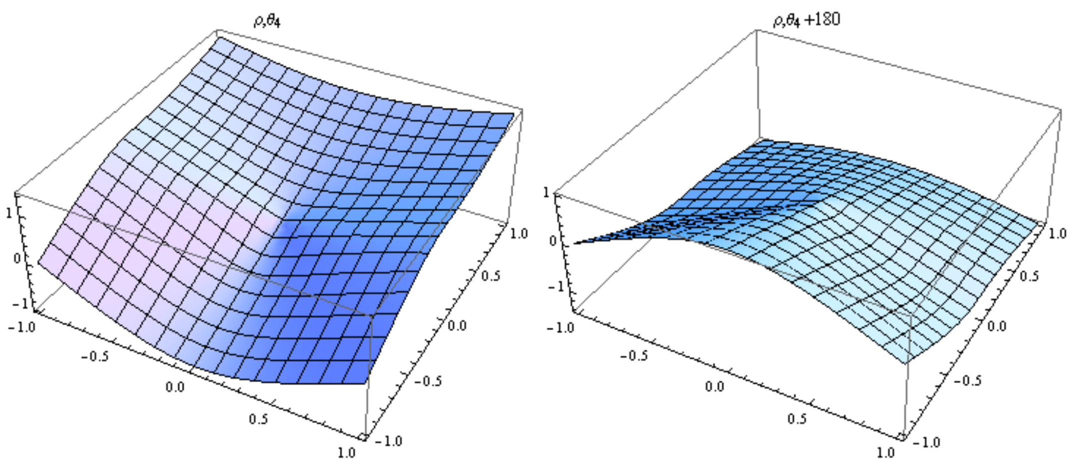
Theorem 3. Let θ_{max} and θ_{min} be the angles at which the angular correlations are maximized and minimized, respectively. Then

a. $\theta_{max} + \theta_{min} = 180^o$

b. For $\rho_{UV} = \rho_{UW} = \rho_{VW} = 1$, $\theta_{max} \in [0^o, 135^o] \cup [315^o, 360^o]$

c. For $\rho_{UV} = \rho_{UW} = \rho_{VW} = -1$, which are incompatible pairwise correlations $\theta_{max} \in \{180^o, 270^o\}$, i.e., $\rho_{U|VW(\theta_{max})} = Cor\{U, -V\} = Cor\{U, -W\}$.

Figures 2 and 3 describe the behavior of extreme angular correlations and direction expressed as radians for a special case. We assumed that two position variables V and W are independent.



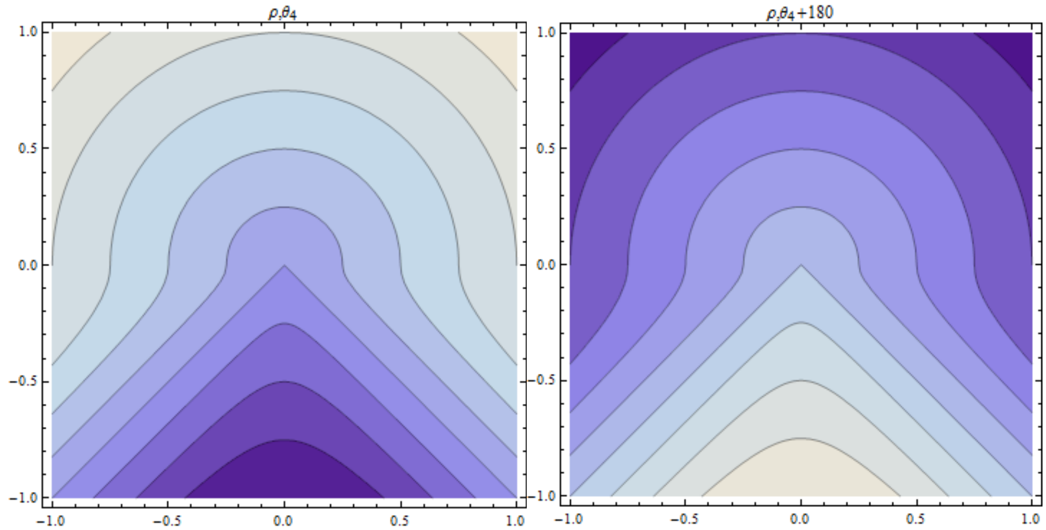
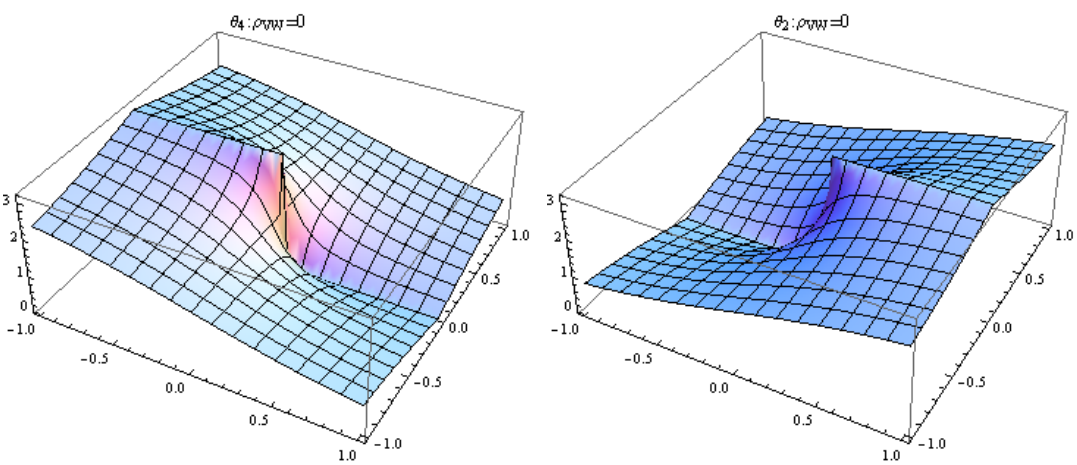


Fig. 2 The correlation plot of $\rho_{U|VW(\theta_{ext}^{4})}$ and $\rho_{U|VW((\theta+180°)_{ext}^{4})}$, respectively, for the case $\rho_{UV}, \rho_{UW}, and\ \rho_{VW} = 0$, along with contour plots of these results, showing the inversion described in Corollary 1.



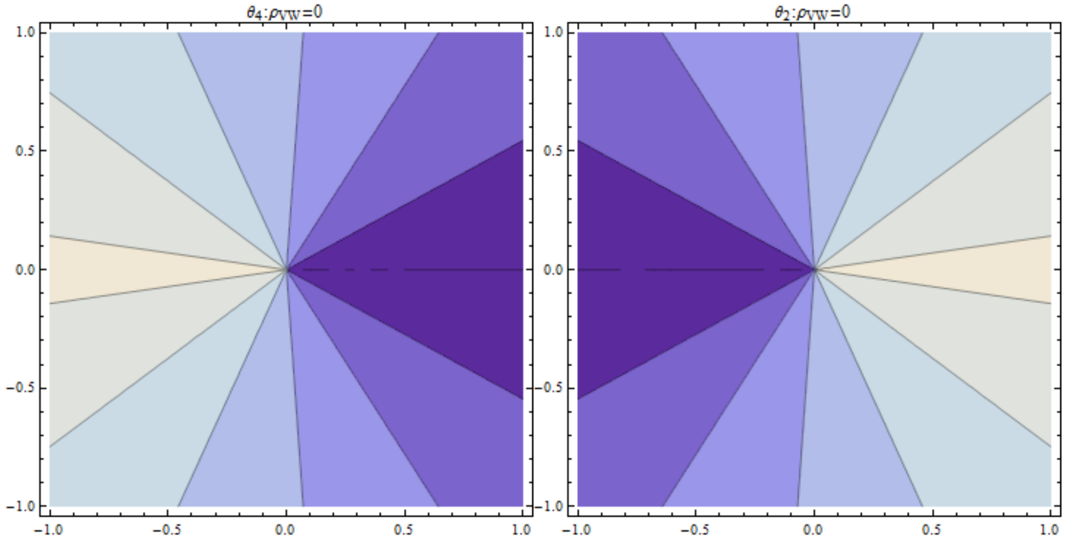


Fig. 3 The correlation plot of θ_{ext}^4 and $(\theta + 180°)_{ext}^4$, respectively, for the case $\rho_{UV}, \rho_{UW}, and\ \rho_{VW} = 0$, along with contour plots of these results, showing the reflection described in Theorem 3.

Corollary 2.

(i) For ρ_{UV}=ρ_{UW}=ρ_{VW}=ρ, the minimum angular correlation of $-\frac{\sqrt{2}\rho}{\sqrt{1+\rho}}$ will be attained at $\theta_{min} = -\frac{3\pi}{4}$. The maximum correlation of $\frac{\sqrt{2}\rho}{\sqrt{1+\rho}}$ will be attained at $\theta_{max} = \frac{\pi}{4}$.

(ii) For ρ_{UV}=$\rho_{UW} = \rho$ and $\rho_{VW} \in [-1,1]$, the minimum angular correlation of $-\frac{\sqrt{2}\rho}{\sqrt{1+\rho_{VW}}}$ will be attained at $\theta_{min} = -\frac{3\pi}{4}$. The maximum angular correlation of $\frac{\sqrt{2}\rho}{\sqrt{1+\rho_{VW}}}$ will be attained at $\theta_{max} = \frac{\pi}{4}$.

(iii) For ρ_{UV}=$\rho_{VW} = \rho$ and $\rho_{UW} \in [-1,1]$, the extreme angular correlations attained at $\theta_{ext} = \pm ArcCos\left[-\frac{\rho(-1+\rho_{UW})}{\sqrt{\rho^2+\rho^4-4\rho^2\rho_{UW}+\rho_{UW}^2+\rho^2\rho_{UW}^2}}\right]$ is

$$\frac{\pm\rho_{UW}\sqrt{\frac{\left(\rho^2-\rho_{UW}\right)^2}{\rho^2+\rho^4-4\rho^2\rho_{UW}+(1+\rho^2)\rho_{UW}^2}}-\frac{\rho^2(-1+\rho_{UW})}{\sqrt{\rho^2+\rho^4-4\rho^2\rho_{UW}+(1+\rho^2)\rho_{UW}^2}}}{\sqrt{1-\rho Sin[2ArcCos\left[\frac{\rho-\rho\rho_{UW}}{\rho^2+\rho^4-4\rho^2\rho_{UW}+(1+\rho^2)\rho_{UW}^2}\right]]}}$$

(iv) For ρ_{UW}=$\rho_{VW} = \rho$ and $\rho_{UV} \in [-1,1]$, the extreme angular correlations attained at $\theta_{ext} = \pm ArcCos\left[-\frac{\rho^2-\rho_{UV}}{\sqrt{\rho^2+\rho^4-4\rho^2\rho_{UV}+\rho_{UV}^2+\rho^2\rho_{UV}^2}}\right]$ is

$$\frac{\pm\rho\sqrt{\frac{\rho^2\left(-1+\rho_{UV}\right)^2}{\rho^2+\rho^4-4\rho^2\rho_{UV}+(1+\rho^2)\rho_{UV}^2}}+\frac{\rho_{\square V}(-\rho^2+\rho_{UV})}{\sqrt{\rho^2+\rho^4-4\rho^2\rho_{UV}+(1+\rho^2)\rho_{UV}^2}}}{\sqrt{1-\rho Sin[2ArcCos\left[\frac{-\rho^2+\rho_{UV}}{\rho^2+\rho^4-4\rho^2\rho_{UV}+(1+\rho^2)\rho_{UV}^2}\right]]}}.$$

4. AN APPLICATION

The developed approach to directional dependence is applied to a data set consisting of three random variables for 87 Minnesota counties; total population *(X)*, unemployment rate *(Y)*, and education level *(Z)*. Let $(X_i, Y_i, Z_i)_{i=1,\dots,n}$ be a random sample of size n from the random vector (X, Y, Z), and $(x_i, y_i, z_i)_{i=1,\dots,n}$ be its particular realization. Original variables need to be transformed to uniform distribution individually, leading to a random sample $(U_i, V_i, W_i)_{i=1,\dots,n}$ and its particular realization $(u_i, v_i, w_i)_{i=1,\dots,n}$. At this point we have considered two alternatives:

(i) Introduce assumptions on the univariate marginal distributions, i.e. specify F_X, F_Y, and F_Z, and define $u_i = F_X(x_i)$, $v_i = F_Y(y_i)$, and $w_i = F_Z(z_i)$.

(ii) Define $u_i = (1/n)\sum_{j=1}^{n} \mathbf{1}(x_j \leq x_i)$, $v_i = (1/n)\sum_{j=1}^{n} \mathbf{1}(y_j \leq y_i)$, and $w_i = (1/n)\sum_{j=1}^{n} \mathbf{1}(z_j \leq z_i) = (1/n)rank(z_i)$, where $\mathbf{1}(A)$ is the indicator of event A.

In this research we have selected the second alternative. To understand the directional dependence between these variables, we will consider three sets of position variables (V, W), (U, W), and (U, V) and analyze the angular correlation structure with the remaining variable. Note that, it is straightforward to create a sample version of the angular correlations that we have introduced by using estimated Pearson's correlations from the data $(u_i, v_i, w_i)_{i=1,\ldots,n}$. The estimated angular correlations will be denoted by $r_{U|VW(\theta)}, r_{V|UW(\theta)}$, and $r_{W|UV(\theta)}$. As an example

$$r_{U|VW(\theta)} = \frac{\cos(\theta)\, r_{UV} + \sin(\theta) r_{UW}}{\sqrt{1 + \sin(2\theta) r_{VW}}},$$

where r_{UV}, r_{UW}, and r_{VW} are the estimated Pearson's correlations from the data $(u_i, v_i, w_i)_{i=1,\ldots,n}$.

For this data set the sample Pearson's correlations for total population-unemployment rate, total population-education level, and unemployment rate-education level are 0.167956, 0.592865, and -0.004413, respectively. Figure 4 presents sample angular correlation plots for the three sets of position variables. Table 1 summarizes the results.

Position Pair	*(Unemployment, Education)*	*(Population, Education)*	*(Population, Unemployment)*

Extreme Correlations	± 0.616915	± 0.211857	± 0.602177
Extreme Angles in radians (degrees)	1.29099 (74^o) -1.85061 (-106^o)	-0.547479 (-31^o) 2.59411 (149^o)	-0.17342 (-10^o) 2.96817 (170^o)

Table 1. Extreme sample angular correlations and the angles that they have been attained for the case where $\rho_{UV} = \rho_{UW} = \rho_{VW} = \rho$.

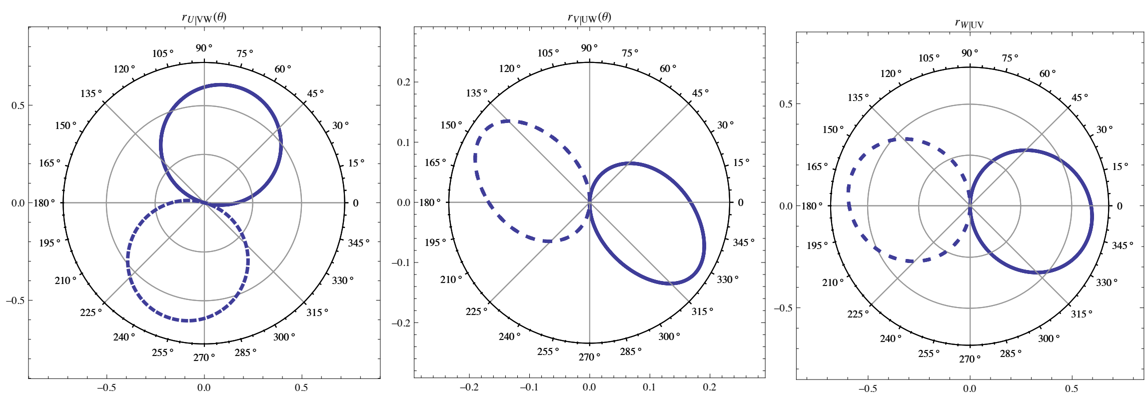


Fig. 4 Sample angular correlation plots for the three sets of position variables, for the case where $\rho_{UV} = \rho_{UW} = \rho_{VW} = \rho$.

The plot on the left shows the angular correlations for the position pair *(Unemployment, Education)*. Here the maximum correlation is 0.616915, attained at (74^o). The middle plot shows the angular correlations for the position pair *(Population, Education)*. The maximum correlation in this set-up is 0.211857, attained at (329^o). The plot on the right shows the angular correlations for the position pair *(Population, Unemployment)*. The maximum correlation of 0.602177 is attained at (350^o). It is clear that the dependence structure changes as one moves from one set of position variables to another.

The results derived from our angular correlation approach for the case where $\rho_{UV} = \rho_{UW} = \rho_{VW} = \rho$ are directly related to canonical correlation analysis. Table 2 shows the results on canonical correlation analysis done on the data $(u_i, v_i, w_i)_{i=1,\ldots,n}$. Note these are the same values obtained in Table 1 by calculating the angular correlations.

Position Pair	*(Unemployment, Education)*	*(Population, Education)*	*(Population, Unemployment)*
Canonical Correlation	0.616915	0.211857	0.602177

Table 2. Canonical correlations for the different position pairs for the case where $\rho_{UV} = \rho_{UW} = \rho_{VW} = \rho$.

5. CONCLUDING REMARKS

Modeling directional dependence is a challenging task. In this research we have introduced an approach that is based on ideas from spatial statistics. The introduction of an angular directional parameter helps to understand some hidden dependence information and also gives a chance to produce meaningful dependence plots.

At the next stage, we aim to study further properties of such measures and provide a similar framework for the dimensions higher than three. Setting up a prior on angular direction parameter θ, such as von Mises distribution would provide a basis to carry out Bayesian inference on modeling directional dependence.

ACKNOWLEDGMENT

The authors greatly appreciate the generous comments and suggestions of two anonymous referees on the first version of this article.

REFERENCES

[1] Clark, D., Understanding canonical correlation analysis, *Concepts and Techniques in Modern Geography,* **3** (1975), 3-33

[2] Dodge, Y., and Rousson, V., Direction dependence in regression line, *Communications in Statistics: Theory and Methods,* **29** (2000), 1957-1972

[3] Nelsen, R. B., *An Introduction to copulas* (2^{nd} edn.), New York: Springer, 2006

[4] Nelsen, R. B., Úbeda-Flores, M., Directional Dependence in Multivariate Distributions. *Annals of the Institute of Statistical Mathematics*, (2011), 1-9.

[5] Schmid, F., Schmidt, R., Multivariate extensions of Pearson's rho and related statistics. *Statistics and Probability Letters,* **77** (2007), 407-416.

[6] Simon, G., An angular version of spatial correlation, with exact significance tests. *Geographical Analysis,* **29** (1997), 267-278

[7] Sklar, A. Functions de repartition à *n* dimensions et leurs marges, *publications de l'Institut de Statistique de l'Université de Paris,* **8** (1959), 229-231

[8] Sungur, E.A., A Note on directional dependence in regression setting, *Communications in Statistics: Theory and Methods,* **34** (2005), 1957-1965

[9] Sungur, E.A., and Celebioglu. S., Copulas with directional dependence property, *Gazi University Journal of Science,* **24(3)** (2011), 415-424

[10] Tishler, A. and S. Lipovetsky, "Canonical Correlation Analysis for Three Data Sets: A Unified Framework with Applications to Management", *Computers and Operations Research*, **23** (1996), 667-679.

[11] Tishler A., and Lipovetsky S., “Modelling and Forecasting with Robust Canonical Analysis: Method and Application”, *Computers and Operations Research*, **27** (2000), 217-232.