II. Chapter 2

Random Variables

II.1. Review

For this chapter we need to know some mathematical facts on the functions. These facts will help us to understand the properties of the probabilistic models that we are going to develop. A **function** f from a set D to a set E is a correspondence that assigns to each element of D a unique element of E.

Exercise 1. Suppose that coin tossed three times, leading to a sample space S={HHH,HTT,THT,TTH,THH,HTH,HHT,TTT}. Define X=Number of heads observed when we tossed a coin three times. Is X a function? If yes, what is D and E?

Discrete Mathematics

To be able to understand and derive the properties of the discrete probabilistic models we need the following facts from discrete mathematics.

Binomial Theorem: The quantity $(a+b)^n$ can be expressed in the form

$$(a+b)^n = \sum_{i=0}^{n} \binom{n}{i} a^i b^{n-i} ,$$

where

$\binom{n}{i} = \frac{n!}{i!(n-i)!}$; i=0 , ..., n.

For example, we have

$(a+b)^0 = 1$

$(a+b)^1 = a+b$

$(a+b)^2 = a^2 +2ab+b^2$

$(a+b)^3 = a^3 +3a^2 b+3ab^2 +b^3$.

Note: In probability we will use binomial theorem not to extend $(a+b)^n$ but to evaluate the sums of the form $\sum_{i=0}^{n} \binom{n}{i} a^i b^{n-i}$. For example can you figure out the value of the following sum?

$$\sum_{i=0}^{n} \binom{n}{i} p^i (1-p)^{n-i} = ? =$$

Hint: a=?= , b=?=

What about the following sum?

$$\sum_{i=0}^{n} \binom{n}{i} n\, p^i (1-p)^{n-i} = ? =$$

Maybe we should look for a better way of evaluating this sum?

Geometric Series:

Let $a \neq 0$.

$$\sum_{i=0}^{\infty} ar^i = a+ar+ar^2 +ar^3 +...+ar^i +...$$

is called geometric series which

(i) converges and has the sum $\frac{a}{1-r}$ if $|r|<1$.

(ii) diverges if $|r|\geq 1$.

In probability geometric series is used in a discrete distribution which is called geometric distribution. Can you find the value of the following sum?

$$\sum_{n=1}^{\infty}(1-p)^{n-1}p \ = ? =$$

Note that the lower limit of the sum starts from 1 not 0 as in the geometric series. Maybe we should define i=n-1. Then, what is a=?= , r=?=

Maclaurin Series:

One of the Maclaurian series that is used in the probability is the following

$$e^x = \sum_{i=0}^{\infty}\frac{a^i}{i!} = 1+ a + \frac{a^2}{2!} + \frac{a^3}{3!} + \ldots .$$

By using the Maclaurian series can you find the value of the following sum?

$$\sum_{i=0}^{\infty} e^{-\lambda}\frac{\lambda^i}{i!} = ? =$$

What about the following sum which is used to find expected value of a Poisson random variable.

$$\sum_{i=0}^{\infty} i\, e^{-\lambda}\frac{\lambda^i}{i!} = ? =$$

Note that $\frac{i}{i!} = \frac{1}{(i-1)!}$, so take one of the λ^i out of the summation and ...

The Derivative of a Function:

One way of understanding the behavior of a function f(x), is to look at its derivative. With each function f(x) we associate another function, called "the derivative of f(x)". This derivative is a formula for the rate at which f(x) is changing as x changes, and it measures the steepness of the graph of f(x). Here are the some of the formulas for the derivatives that you should know(g(x) and h(x) are differentiable functions and a, and b are arbitrary constants):

function f(x)	**its derivative** $f'(x) = \frac{df(x)}{dx}$
$ax+b$	a
$ax^n + b$	nax^{n-1}
$e^{ax} + b$	ae^{ax}
$e^{-ax} + b$	$-ae^{-ax}$
$e^{g(x)} + b$	$g'(x)\, e^{g(x)}$
$a\ln(x)+b$	$\frac{a}{x}$
$h(x)g(x)$	$h'(x)\, g(x)+h(x)g'(x)$
$\frac{h(x)}{g(x)}$	$\frac{h'(x)g(x) - g'(x)h(x)}{[g(x)]^2}$

Table 1. Table of Derivatives for Some Functions.

A derivative of an integral as a function of its limit of integration can be found by using the following relation (f(x) is a "nicely behaved" function)

$$\frac{d}{dx}\int_a^x f(t)dt = f(x).$$

Derivatives of the Composite Functions: The Chain Rule

If f=f(u) is a function of input variable u and u=u(x) is a function of the input variable x, then

$$\frac{d}{dx} f(u(x)) = \frac{df(u)}{du}\frac{du}{dx} .$$

This rule is useful in finding derivatives of functions with a complicated relationship.

Exercise 2. Find the derivatives of the following functions

(a) $F(x)=1 - \lambda e^{-\lambda x}$

(b) $M(t)=\frac{\lambda}{\lambda - t}$

(c) $M(t)=\frac{e^{tb} - e^{ta}}{t(b - a)}$

(d) $F(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}\sigma} e^{-(z-\mu)^2/2\sigma^2} dz.$

The Definite Integral of a Function over an Interval: The definite integral of a function over an interval, say $a \le x \le b$, of x-values, is a number which measures the amount of f(x) aggregated in that interval.

Facts:

(i) If f(a) exists, then $\int_a^a f(x)dx = 0.$

(ii) If f is integrable and $f(x) \ge 0$ for every x in [a,b], then the area A of the region under the graph of f from a to b is

$$A = \int_a^b f(x)dx .$$

(iii) For a < c < b

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx.$$

(iv) If $f(x) \ge 0$ and integrable on [a,b], then

$$\int_a^b f(x)dx \ge 0.$$

Antidifferentiation is the reverse process to differentiation, that is antidifferentiation is the process of finding a formula for a quantity if you its rate of change. The process of antidifferentiation is commonly called **integration** or **indefinite integration**. The absence of the values bounding x at the top and bottom of the integration symbol indicate that it calls for antiderivative and not the definite integral.

$\int u dv = uv - \int v du$
$\int u^n du = \frac{1}{n+1} u^{n+1} + C, n \neq -1$
$\int \frac{du}{u} = \ln\lvert u \rvert + C$
$\int e^u du = e^u + C$
$\int a^u du = \frac{1}{\ln a} a^u + C$
$\int \frac{du}{a^2 + u^2} = \frac{1}{a} \tan^{-1}\frac{u}{a} + C$
$\int \frac{du}{a^2 - u^2} = \frac{1}{2a} \ln \left\lvert \frac{u+a}{u-a} \right\rvert + C$
$\int \frac{du}{u^2 - a^2} = \frac{1}{2a} \ln \left\lvert \frac{u-a}{u+a} \right\rvert + C$

Table 2. Table of Integrals

Evaluation of the Indefinite Integrals: By reversing the table of derivatives (Table.1), we obtain a list of antiderivatives of many basic types of functions. More complicated functions need to be re-expressed in a simpler form by algebraic manipulations or substitutions.

Evaluation of the definite integral by using antiderivatives can be done by taking the difference in the values of an antiderivative of f(x) (if one can be determined) at x=b and x=a. That is

$$\int_a^b f(x)dx = F(x)\Big|_a^b = F(b) - F(a)$$

where,

$$F(x) = \int f(x)dx\ .$$

Note that for the definite integral the constant in the antiderivative disappears.

The Three Elementary Methods of Integration. Given an integral to evaluate whose formula is not in the table of integrals: what to do? Often we can obtain the result we seek by using one of the three elementary methods of integration from the calculus. Do you remember them? 1) Ask your room-mate; 2) ask the instructor; and 3) look it up. That is only partly facetious. There are extensive tables of integrals in the CRC-Handbook and in other sources, and "Mathematica" is a powerful piece of computer software that will readily evaluate integrals far more difficult than anything that we are likely to encounter. Nonetheless, the three elementary methods are useful tools with which we should be familiar. They are 1) integration by substitution; 2) integration by parts; and 3) integration by partial fractions. In this course we will mostly use the first two methods which are explained below.

Integration by Substitution.
Indefinite Integral

I will explain the method for the following trivial integral,

$$\int \frac{dx}{3x+5}$$

Step 1. Find a u=u(x) such that when we plug in u the unfamiliar integral will transform into one we recognize.
Step 2. Find the relation between dx and du, and replace former by the later.
Step 3. Find the integral (antiderivative) of the recognizable form.
Step 4. Plug in u(x) in your result.

We know that $\int \frac{du}{u} = \ln|u| + C$, so reasonable choice for

(recognize u(x))

$u(x) = 3x+5$, and

(relation between dx and du)

$$du=3dx \Rightarrow dx=\frac{1}{3}\ du$$

(replace former by the later, and find antiderivative)

$$\int \frac{1}{3}\frac{1}{u}\ du = \frac{1}{3}\int \frac{1}{u}du\ = \frac{1}{3}\ \ln|u| + C$$

(put back u(x))

$$\int \frac{dx}{3x+5} = \frac{1}{3}\ \ln|3x+5| + C$$

Definite Integral

I will explain the method for the following trivial integral,

$$\int_0^2 \frac{dx}{3x+5}$$

Step 1. Find a u=u(x) such that when we plug in u the unfamiliar integral will transform into one we recognize.
Step 2. Find the relation between dx and du, and replace former by the later.
Step 3. Find the new limits in terms of u.
Step 4. Find the integral (antiderivative) of the recognizable form.
Step 5. Find the value of the definite integral.

We know that $\int \frac{du}{u}$ = ln|u| + C, so reasonable choice for

(recognize u(x))

u(x) = 3x+5, and

(relation between dx and du)

$$du=3dx \Rightarrow dx=\frac{1}{3}\ du$$

(replace former by the later, find new limits, and find antiderivative)

$$\int_0^2 \frac{dx}{3x+5} =$$

$$\int_5^{11} \frac{1}{3}\frac{1}{u}\ du = \frac{1}{3}\int_5^{11} \frac{1}{u\ du} = \frac{1}{3}\ \{\ln 11 - \ln 5\} = \frac{1}{3}\ \ln\frac{11}{5}$$

Integration by Parts.

(Attention: This is the method that we are going to use most frequently)

This method is based on the simple rule for finding the differential of a product:

$$d(uv) = vdu + udv$$

Rearranging and integrating,

$$\int udv\ = uv - \int vdu$$

it being understood that if these are not definite integrals, we must add an integration constant. We are initially stumped by the integral on the left, but in our mind's eye we see a way to rearrange the integral into two parts such that we can evaluate the integral on the right.

Here is an example and the steps that you need for this method:

integral	$\int xe^x\ dx$
define u and dv	u=x and dv=e^x dx *(Note that we know how to find the antiderivative of dv)*
find du and v	du=1dx and v=$\int e^x dx\ = e^x$
plug in u, du, v, and dv into the formula for integration by parts $\int udv\ = uv - \int vdu$	$\int xe^x\ dx = xe^x\ - \int (1)\ (e^x)\ \ dx$
and the result	$\int xe^x\ dx = xe^x\ - e^x\ + C$

For some problems it is necessary to repeatedly use integration by parts. For such type of problems it is more useful to produce reduction formulae. For example using integration by parts we can establish that

$$\int x^n\ e^{ax}dx\ = \frac{1}{a}\ x^n\ \ e^{ax}\ - \frac{n}{a}\int x^{n-1}\ e^{ax}\ dx$$

This formula can be used repeatedly until all powers of x have been removed.

Some Results Involving Limits

In this section, we simply remind you of some of the techniques on the calculations of the limits that we find most useful in probability theory. These techniques will be useful on understanding Central Limit Theorem, and finding moments by using moment generating functions.

Early in a calculus course the existence of the following limit is discussed and is denoted by the letter e:

$$e = \lim_{t\to 0}(1+t)^{1/t} = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{n}$$

Often it is rather easy to see the value of certain limits. For example, with -1 < r < 1, the sum of the geometric progression allows us to write

$$\lim_{n\to\infty}(1+\rho+\rho^2+\rho^3+\ldots+\rho^n) = \lim_{n\to\infty}\left(\frac{1-\rho^n}{1-\rho}\right) = \frac{1}{1\text{-r}} .$$

Since $\lim_{n\to\infty} r^n = 0$, the limit of the ratio can be easily found.

However it is not that easy to determine the limit of every ratio; for example consider

$$\lim_{b\to\infty}(be^{-b}) = \lim_{b\to\infty}\left(\frac{b}{e^b}\right).$$

Since both the numerator and the denominator of the latter ratio are unbounded, we will not be able to find the limit easily.

Consider the ratio f(x)/g(x), if the limits of f(x) and g(x) have the limits ∞ or -∞ as x approaches to c, we say that f(x)/g(x) has the **indeterminate form ∞/∞** at x=c. If the limits of both functions approaches to 0, then we say that f(x)/g(x) has the **indeterminate form 0/0** at x=c. **L'Hospital's Rule** can be applied to the indeterminate forms to find the limit of the ratio.

L'Hopital's Rule: Suppose the functions f and g are differentiable on an open interval (a,b) containing c, except possibly at c itself. If g'(x) ≠ 0 for x ≠ c, and if f(x)/g(x) has the intermediate form 0/0 or ∞/∞ at x=c, then

$$\lim_{x\to c}\frac{f(x)}{g(x)} = \lim_{x\to c}\frac{f'(x)}{g'(x)}$$

provided that $\lim_{x\to c}$ [f'(x)/g'(x)] exists or $\lim_{x\to c}$ [f'(x)/g'(x)] = ∞.

Therefore, $\lim_{b\to\infty}\left(\frac{b}{e^b}\right)$ can be found taking the limit of the ratio of the derivative of the numerator and the derivative of the denominator. We have

$$\lim_{b\to\infty}\left(\frac{b}{e^b}\right) = \lim_{b\to\infty}\left(\frac{1}{e^b}\right) = 0.$$

Note that you can use L'Hopital's rule more than once.

Some Results Involving Multivariate Calculus

In this section we only make some suggestions about functions of two variables, say

$$z = g(x,y)).$$

But these results can be extended to more than two variables. Usually all the results that we have learned for the one variable case can be used for the two variable case by treating the "other" variable as a constant. The two *first partial derivatives* with respect to x and y, denoted by $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ can be found in the usual manner of differentiating by treating the "other" variable constant. For illustration consider the following simple examples,

$$z = f(x,y) = xy$$

$$\frac{\partial z}{\partial x} = \frac{\partial f(x,y)}{\partial x} = \frac{\partial (xy)}{\partial x} = y$$

and

$$\frac{\partial z}{\partial x} = \frac{\partial (e^{xy^2})}{\partial x} = (e^{xy^2})(2xy).$$

The *second partial derivatives* are simply first partial derivatives of the first partial derivatives. If z=xy, then

$$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial x}(x) = 1.$$

For notation we use

$$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial^2 z}{\partial x^2},$$

$$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial^2 z}{\partial x \partial y},$$

$$\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial^2 z}{\partial y \partial x},$$

$$\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial^2 z}{\partial y^2}.$$

In general,

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial x},$$

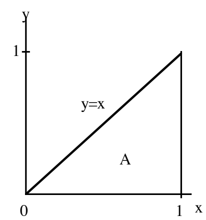
provided that partial derivatives involved are continuous functions.

The value of the double integral

$$\iint_A f(x,y)dxdy$$

can usually be evaluated by an iterated procedure; that is, evaluating two successive single integrals. For illustration, say $A = \{(x,y): 0 \le x \le 1, 0 \le y \le x\}$.
Then

When placing the limits on the iterated integral, note that for each fixed x between zero and one, y is restricted to the interval zero to x (see the figure below). Also in the inner integral on y, x is treated as a constant.



In evaluating this double integral we could have restricted y to the interval zero to one, then x would be between y and one. That is, we would have evaluated the iterated integral

$$\iint_A f(x,y)dxdy = \iint_A (x + x^3y^2)dxdy$$

$$= \int_0^1 \left[\int_y^1 (x + x^3y^2)dx\right]dy$$

$$= \int_0^1 \left[\frac{x^2}{2} + \frac{x^4y^2}{4}\right]_y^1 dy$$

$$= \int_0^1 \left(\frac{1}{2} + \frac{y^2}{4} - \frac{y^2}{2} - \frac{y^6}{4}\right)dy = \left[\frac{y}{2} - \frac{y^3}{3\cdot 4} - \frac{y^7}{7\cdot 4}\right]_0^1$$

$$= \frac{1}{2} - \frac{1}{12} - \frac{1}{28} = \frac{8}{21}.$$

Finally, we will look at *change of variables in a double integral*

$$\iint_A f(x_1,x_2)dx_1dx_2$$

If $f(x_1, x_2)$ is a joint probability density function of X_1 and X_2, then the above double integral represents $P[(X_1, X_2) \in A]$. Consider only one-to-one transformations, say $y_1 = g_1(x_1,x_2)$ and $y_2 = g_2(x_1,x_2)$ with inverse transformations given by $x_1 = h_1(y_1,y_2)$ and $x_2 = h_2(y_1,y_2)$. The determinant of order 2

$$J(x_1,x_2) = \begin{vmatrix} \frac{\partial g_1}{\partial x_1} & \frac{\partial g_1}{\partial x_2} \\ \frac{\partial g_2}{\partial x_1} & \frac{\partial g_2}{\partial x_2} \end{vmatrix} = \frac{\partial g_1}{\partial x_1}\frac{\partial g_2}{\partial x_2} - \frac{\partial g_1}{\partial x_2}\frac{\partial g_2}{\partial x_1} \neq 0$$

is called the **Jacobian** of the transformation. And

$$J(y_1,y_2) = \begin{vmatrix} \frac{\partial h_1}{\partial y_1} & \frac{\partial h_1}{\partial y_2} \\ \frac{\partial h_2}{\partial y_1} & \frac{\partial h_2}{\partial y_2} \end{vmatrix} = \frac{\partial h_1}{\partial y_1}\frac{\partial h_2}{\partial y_2} - \frac{\partial h_1}{\partial y_2}\frac{\partial h_2}{\partial y_1} \neq 0$$

is called the Jacobian of the inverse transformation. Moreover, say the region A maps onto B in the (y_1, y_2) space. Since we are usually dealing with probabilities in this course, we fixed the sign of the integral so that it is positive by using the absolute value of the Jacobian. Then it is true that

$$\iint_A f(x_1,x_2)dx_1dx_2 = \iint_B f\left[h_1(y_1,y_2), h_2(y_1,y_2)\right]\left|J(y_1,y_2)\right|dy_1dy_2.$$

That is $P[(X_1, X_2) \in A] = P[(Y_1, Y_2) \in B]$.

Note that $|J(y_1, y_2)| = |J(x_1, x_2)|^{-1}$, but the right side is often easy to obtain, rather than solving for x_1 and x_2 in terms of y_1 and y_2 or differentiating implicitly.

Now, let us look at the steps required for the multivariate change of variables on the integration.

$f(x_1, x_2)$	$f(x_1,x_2) = e^{-(x_1+x_2)}$
DETERMINE THE REGION A	$A = \{(x_1,x_2): 0 < x_1 < \infty, 0 < x_2 < \infty\}$
DETERMINE g_1 AND g_2	$g_1(x_1,x_2) = x_1$ $g_2(x_1,x_2) = x_1 + x_2$
DETERMINE h_1 AND h_2	$h_1(y_1,y_2) = y_1$ $h_2(y_1,y_2) = y_2 - y_1$
DETERMINE THE REGION B	$B = \{(y_1,y_2): 0 < y_1 < y_2 < \infty\}$
FIND THE JACOBIAN, J	$J = \begin{vmatrix} \frac{\partial}{\partial x_1}(x_1) & \frac{\partial}{\partial x_2}(x_1) \\ \frac{\partial}{\partial x_1}(x_1+x_2) & \frac{\partial}{\partial x_2}(x_1+x_2) \end{vmatrix}$ $= \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} = 1 \bullet 1 - 0 \bullet 1 = 1$
FIND THE ABSOLUTE VALUE OF THE JACOBIAN, $\|J\|$	$\|J\| = \|1\|$ $\|J(y_1,y_2)\| = \frac{1}{\|J\|} = 1$
WRITE DOWN THE TRANSFORMED INTEGRAL	$\int_0^\infty\int_0^\infty e^{-(x_1+x_2)}dx_1dx_2 = \int_0^\infty\int_{y_1}^\infty e^{-[y_1+(y_2-y_1)]} \cdot 1 \cdot dy_2dy_1$

Some Special Functions

Note that our objective is to develop probabilistic models for the characteristic that we are interested with, such as survival time of a cancer patient, life time of a TV set, or simply the midterm exam scores of Math. 3610 students. Since the behavior of all the functions are not the same the functional form of the model will characterize the random variable differently. We will discuss this in detail in the next section. Now let us look at some special functions and their properties.

Exponential Function

The function $f(x)=e^x$ is called exponential function (frequently written exp(x)).

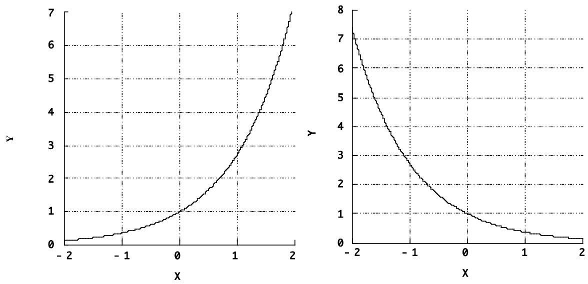


Figure 1. Graphs of e^{x} and e^{-x} .

Note that $e^{0}=1$ and $e^{-\infty}=0$.

$e^{a}e^{b}=e^{a+b}$

$(e^{a})^{n}=e^{na}$

Gamma Function

The gamma function, denoted by $\Gamma(\alpha)$ for all $\alpha>0$, is given by

$$\Gamma(\alpha)=\int_0^{\infty} e^{-x}x^{\alpha-1}dx$$

The gamma function satisfies the following properties:

$\Gamma(\alpha)=(\alpha-1)\Gamma(\alpha-1) \quad \alpha>1$

$\Gamma(n)=(n-1)!=(n-1)(n-2)...(1) \qquad n=1, 2, \ldots$

(remember that O!=1 so $\Gamma(1)=1$)

$\Gamma(\frac{1}{2})=\sqrt{\pi}$.

Beta Function

For $\alpha>0$ and $\beta>0$, the beta function is given by

$$\mathrm{beta}(\alpha,\beta)=\int_0^1 x^{\alpha-1}(1-x)^{\beta-1}dx$$

The beta function has the following properties:

$\mathrm{beta}(\alpha,\beta)=\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$

$\mathrm{beta}(\alpha,\beta)=\mathrm{beta}(\beta,\alpha)$

Exercise 3.

The following type of problems which involves integration in distribution theory to develop probabilistic models:

(a) Find the value of k such that

$$\int_1^{\infty} kx^{-(k+1)}dx = 1.$$

Answer: k>0.

(b) Find the value of c such that

$$\int_0^1 c(1-x)x^2dx = 1.$$

Answer: c=12.

(c) Find the value of c such that

$$\int_0^1 cx^3(1-x)^5 dx = 1.$$

Hint: Use the definition of the beta function.
Answer: 20.

The following integrals arises in finding distribution functions given probability density functions.

(d) Find the value of the following integral

$$F(x) = \int_1^x kw^{-(k+1)} dx .$$

(F(x) is called cumulative distribution function.)
Answer: $1-x^{-k}$; $1<x<\infty$.

(e) Find the value of the following integral

$$F(x) = \int_0^x \lambda e^{-\lambda w} dx .$$

Answer: $1 - e^{-\lambda x}$; $x>0$.

The following integrals arises when we want to find expected values.

(f) Find the value of the following integral

$$E(X) = \int_0^{\infty} x\lambda e^{-\lambda x} dx .$$

(The value of this integral is known as the first moment or expected value of X)
Answer: $\frac{1}{\lambda}$.

(g) Find the value of the integral

$$E(X^2) = \int_0^{\infty} x^2\lambda e^{-\lambda x} dx .$$

Answer: $\frac{2}{\lambda^2}$.

The following type of integrals arises when we want to find the moment generating functions.

(h) Evaluate the following integral

$$\phi_X(t) = \int_0^{\infty} e^{tx}\lambda e^{-\lambda x} dx \; ; \; \lambda>0$$

($\phi_X(t)$ is called moment generating function of exponential distribution with parameter λ.)
Answer: $\frac{\lambda}{\lambda - t}$.

(i) Evaluate the following integral

$$\phi_X(t) = \int_a^b \frac{e^{tx}}{b-a} dx .$$

($\phi_X(t)$ is the moment generating function of a uniform random variable over the interval (a,b)).
Answer: $\frac{e^{tb} - e^{ta}}{t(b - a)}$.

(j) Find the derivative of the $\phi_X(t)$ in (i) with respect to t.

Answer: $\frac{(be^{tb} - ae^{ta})t - (e^{tb} - e^{ta})}{t^2(b - a)}$

(k) Find $\phi_X(0)$ for (j).

(l) Let $y_1 = x_1 - x_2$ and $y_2 = x_1 + x_2$. Find $J(x_1, x_2) = J$, and $J(y_1, y_2)$

Answer: $J(y_1, y_2) = 1/2$, $J(x_1, x_2) = 2$

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